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THE

# MATHEMATICAL GAZETTE.

EDITED BY

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WITH THE CO-OPERATION OF

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## The Mathematical Association.

THE Annual Meeting of the Mathematical Association was held at the London Day Training College, Southampton Row, London, W.C. 1, on Wednesday, 9th January, 1918, at 5.30 p.m., and Thursday, 10th January, at 11.0 a.m. and 2.30 p.m.

*WEDNESDAY*, 5.30 *p.m.*

(1) **ADVANCED SECTION**: "The Graphical Treatment of Power-Series," by Dr. W. P. Milne.

*THURSDAY*, 11 *a.m.*

(2) The following Report of the Council for the year 1917 was distributed, taken as read, and adopted:

During the year 1917, 46 new members have been elected, and the number of members now on the Roll is 653. Of these 8 are honorary members, 38 are life members by composition, 36 are life members under the old rule, and 571 are ordinary members. The number of associates is about 170.

The Council regret to have to record the deaths of Dr. W. H. Besant, F.R.S., *Major G. M. Bell, D.S.O. (of Winchester College), Rev. W. Done Bushell (formerly of Harrow), Mr. Edward Carlisle, Miss A. J. Cooper, *Instructor W. E. Hartley, R.N. (formerly of the Observatory, Cambridge), Mr. F. G. Taylor (of University College, Nottingham), and Rev. A. L. Watherston. Those whose names are marked with an asterisk lost their lives whilst serving with His Majesty's Forces. Mr. Hartley was in H.M.S. *Vanguard*. Dr. Besant was elected an honorary member in 1882, and Mr. Bushell was one of the founders of the Association in 1871.

The Council again recommend that the regulations which govern the election of the Teaching Committees be temporarily suspended, and that the term of office of the existing Committees be extended for one year. It is very difficult to arrange meetings in these times, and the cost of an election would be very considerable.

The Report on the Correlation of Mathematical and Science Teaching has been reprinted, and can be obtained (price 6d.) from the Publishers, Messrs. G. Bell & Sons, Ltd., or from one of the Honorary Secretaries.

The Report of the Girls' Schools Committee on the Teaching of Mathematics in Girls' Schools was issued to the members of the Association early in the year.

Miss H. M. Sheldon has resigned her seat on the Council. Mr. W. E. Paterson and Mr. C. O. Tuckey retire by rotation, and are not eligible for re-election for the coming year. The members present at the Annual Meeting will be asked to nominate and elect others to fill the vacancies.

The Council again desire to acknowledge the indebtedness of the Association to Mr. Greenstreet for his services as Editor of the *Mathematical Gazette*, and to offer their thanks to the authorities of the London Day Training College for their kindness in affording accommodation for the Annual Meeting, and for the meetings of the Council and of the Committees which have been held during the year.

(3) The Treasurer's Report for the year 1917 was approved.

(4) The Re-election of the Teaching Committees. It was proposed that the existing Teaching Committees should remain in office for another year. The proposal was agreed to.

(5) The vacancies on the Council were filled by the election of Miss E. Glauert, Principal J. L. S. Hatton, and Mr. J. Strahan.

(6) **GENERAL SECTION.** The following papers were read:

(i) "The Uses and Functions of a School Mathematical Library," by Dr. W. P. Milne.

(ii) "Nomography," by Dr. S. Brodetsky.

(iii) "Some Suggestions for a Presentment of Mathematics in closer touch with Reality," by Mr. G. Goodwill.

*THURSDAY*, 2.30 *p.m.*

(iv) The President's Address: "Mathematics and Individuality."

(v) "The Position of Mathematics in the New Scheme of the Board of Education for Secondary Schools." A discussion on this subject was opened by Mr. W. D. Eggar, Mr. P. Abbott, Miss J. Dow, and others.

The following resolution was proposed by Dr. J. L. S. Hatton and carried unanimously:

"That the Mathematical Association hopes that, before the new suggested Regulations for the Examination for Class I. of the Civil Service are carried into effect, it will be made clear that a larger scope will be given to Mathematics in Section A than is suggested by the specimen paper in Appendix C, and further, that the description of Subject 4 in the Scheme of Examination be amended by the addition of the words 'and mathematics.'"

Dr. Hatton also proposed:

"That in the opinion of the Mathematical Association the total marks assigned to Subject 4, if amended in accordance with the foregoing resolution, should be two hundred."

This was carried by a large majority.

## NOTICE.

### *EXAMINATION PAPERS.*

THE General Teaching Committee has appointed a Sub-Committee to report periodically on the current examination papers of public examining bodies, so far as mathematical subjects are concerned. It would help the work of the Sub-Committee if members of the Association would send copies of any such papers as seem to them to call for comment, with their criticisms, to G. W. Palmer, Christ's Hospital, Horsham.

## GLEANINGS FAR AND NEAR.

**11. Canned.** "How Trigonometry and Calculus are canned for the gunner's convenience."

"Such problems [as those of Gunnery] are peculiarly those of canned Mathematics."—J. M. Bird [*Mathematics Teacher*, Sept. 1917, p. 35].

**12. Equimultiples.** "The student of the fifth book of Euclid should remember that [equimultiples] has no singular."—De Morgan [*Equimultiples*, P.C. 1837].

**13. Proper and Improper.** "This absurd distinction [between *proper* and *improper* fractions] is now beginning to be abolished."—De Morgan [*Fractions*, P.C. 1837].

**14. Precocity.** "When still a child so young as to be unacquainted with my letters, I possessed an extraordinary faculty of the boy Biddle kind for figures, and could promptly render an account of arithmetical questions, such as were put to me by the gentlemen who were my father's associates, and receive from them in return expressions of admiration and immense rewards enabling me to scatter blessings around in the shape of gingerbread and sweetmeats. To be treated as a precocious phenomenon is a dangerous shoal, but as my talent left me as strangely as it had arrived, I was not long exposed to it. With the acquisition of the A, B, C, the gift of calculation suddenly departed, and from that hour to this a more unready reckoner than I have been never existed in this world. It has seemed as if all my capacity in this way had been exhausted between my birthday and its fourth anniversary."—Jerdan's *Autobiography* (1852), i. p. 17.

# THE PRESIDENTIAL ADDRESS.

## MATHEMATICS AND INDIVIDUALITY.

By Professor T. P. Nunn, D.Sc.

In the ancient story of *Barlaam and Ioasaph*, which edified and delighted our medieval forefathers through many centuries, there is a charming apologue of a fowler and a nightingale whom the former had snared and was about to slay.* "Free me from my fetters," said the nightingale, "and I will give thee three precepts by the keeping of which thou shalt be greatly benefited all thy life long." The astonished fowler accepted the terms and set the bird free, receiving as his reward the precepts: "Never try to attain to the unattainable; never regret the thing past and gone; and never believe the word that passes belief." But as the bird flew aloft she cried to her captor, "Shame, sir, on thy recklessness! What a treasure thou hast lost to-day! For I have inside me a pearl larger than an ostrich egg." When the fowler heard this he was distraught and would fain have taken her again. "Come hither," he said, "into my house; I will make thee right welcome, and send thee forth with honour." But the nightingale replied, "Now I know thee to be a mighty fool. Though thou didst receive my words readily and gladly, thou hast gained no profit thereby. I bade thee never regret the thing past and gone; and behold thou art distraught with grief because I am escaped out of thy hands. I charged thee not to try to attain to the unattainable; and thou triest to catch me again. Besides which I bade thee never believe a word past belief; and behold thou hast believed that I had in me a pearl exceeding the size of my whole body."

The fable is capable of many useful applications; but the case I have in mind at the moment is that of a President of this Association who is tempted to dwell on his unworthiness to occupy the seat of his distinguished predecessors. Taking warning from the fowler's folly, I reject, as passing belief, the thought that I can be really fit to follow those eminent mathematicians, and yet refrain from deploring that such greatness has been thrust upon me. At the same time I decline to pursue the unattainable end of producing a mathematical address worthy of the traditions of this chair. There is, fortunately, another way open. My immediate predecessor is, in conjunction with Mr. Russell, the author of a mathematical work which a prescient and authoritative critic has saluted as marking an epoch in modern thought. Yet in his truly admirable presidential addresses Prof. Whitehead chose not to speak of the technical studies in which he is so great a master, but to set before us views on education as a whole in which his views on mathematics hold only a subsidiary, though a necessarily important place. I propose to follow this precedent—to follow it not merely because it suits my incapacity, but because I heartily approve of the policy it illustrates.

This society began as an Association for the Improvement of Geometrical Teaching. In the course of time it has widened the range of its activities, but remains true to its original impulse. We confine, and rightly confine, our activities to practical questions that concern us as teachers of mathematics to boys and girls at school. The view we have taken of the proper scope of our corporate activities serves also as a good working ideal for the individual member. The time is past when either Master Tommy Merton or his friend Harry Sandford could be adequately educated by mere companionship even with so highly cultivated and virtuous a preceptor as the Reverend Mr. Barlow, who exuded information, mingled with lofty principles, upon the slightest stimulus and in every conceivable direction. Education

*I condense it from the translation by Woodward and Mattingly in the Loeb Classical Library.

has become a highly technical business, requiring, especially in secondary schools, the co-operation of a corps of experts, each versed in a special kind of pedagogical craftsmanship. A certain limitation and concentration of interest on the part of the individual teacher has become, therefore, a virtue, since it is a condition of the higher efficiency promised by the new system.

I believe in that promise. It would, however, be foolish not to recognise that specialist instruction has its own dangers, some obvious, some more subtle. Among the latter I do not include the tendency of the specialist to deem his subject the only one worthy of serious study. That, it is true, is a common error. But it is not only common; it is vulgar, and no educated person should allow himself to fall into it. But I do include a more generous form of error that springs from the same root. The man who loves his subject, and is conscious of his mastery over the technique of teaching it, is liable, if not to mistake, at least to forget, the proper objective of his efforts. He is in danger of looking upon the pupil as provided in order that the subject may be taught instead of thinking of the teaching as provided in order that the pupil may be educated. Perverting a well-worn Latin tag, we may say that he is tempted *propter docendum perdere causas docendi*. A somewhat similar attitude of mind, magnificently exaggerated, once led a young man to declare in my hearing that the heavens and earth were created to illustrate the principles of Mr. Russell's mathematical philosophy!

If we are candid with ourselves we shall, I think, admit that an inordinate lust for giving instruction may easily drive any of us into regrettable excesses. This is a defect of the qualities of the enthusiastic and efficient teacher to which he is subject in proportion to his enthusiasm and efficiency. It makes more dangerous the common liability of the specialist to lose sight of the wider purpose of his departmental activities. It is desirable, then, if only to correct this inevitable bias, that we should sometimes turn from our special inquiries to consider their relation to the broader educational issues.

If there ever was a time when it was particularly incumbent on us to examine our educational faith and to make sure that our works accord with it, to purify our aims and to renew our inspiration, the present is surely the moment. The nation is at last seriously concerned about the results of its inveterate indifference to education. Even the lords of the Philistines are losing their belief in Dagon and are ready to hear the prophets of a nobler faith. If the teachers of England are not to lose this unprecedented opportunity it behoves them to make sure of themselves; to close their ranks; to harmonise their counsels. Their minds must become clear as to the common goal to be sought beyond the immediate objects they pursue, each in his own class-room or laboratory, and they must make a sincere attempt to coordinate their efforts for its attainment.

It would be foolish to maintain that the task is easy; misleading to pretend that we have already reached that agreement upon fundamentals that can be the only sound basis of a national movement forwards. Apart from the rather discreditable wrangle which we fail to dignify with the name "conflict of studies" there is a deeper lack of unity with regard to the ultimate aims of education. It may be that the dislike of clear thinking which foreign critics regard as innate in the English temperament makes the pursuit of intellectual unity, here as elsewhere, distasteful to us. In any case, we do little more than conceal our differences decently from one another and the public when we proclaim that we have a common aim, and that it is "the formation of character," or "the training of all the powers of the mind," or "*mens sana in corpore sano*," or "a preparation for complete living," and so forth. For these, as Mr. Keatinge has trenchantly pointed out, are "nebulous" generalities whose popularity is due precisely to the happy fact that everyone can give them, within comfortably wide limits, just the concrete interpretation he pleases. And when one examines these concrete interpretations—in institutions, in curricula, in principles of school government,

in methods of instruction—it becomes evident that the success of such attempts to express a universal aim of education is largely illusory. In short, differences in educational thought and practice, both now and in the past, are of such a character that it is impossible to attribute them merely to differences of opinion about the best means to a commonly accepted end, or to regard them as merely different manifestations of a common purpose that must necessarily be pursued in differing ways in differing circumstances. On the contrary, they are seen to spring in great measure from the adoption of substantially different ends and to be the expression of seriously divergent purposes.

This lack of agreement is not inscrutable, and the perception of its cause throws much light on the nature of education. The priest, the doctor, the lawyer, the soldier serve limited and specific needs of man or society. While there may be sharp differences of opinion about the value, the scope and the forms of their activities, there is little room for quarrels about their ultimate aims. But the functions of the educator touch life at every point, and are therefore concerned with it as a whole. It follows that all educational aims, when concrete enough to be of practical value, are correlative to ideals of life, and that the divergence between the former is simply the reflexion of the eternal conflict among the latter. For example, if the "Greek view of life" cannot be reconciled with that of the Puritan reformers, it is idle to look for harmony between the conceptions of education which sprang from them. Moreover, it is not only true that no ideal of life has for long reigned unchallenged over civilised men, even of the same race and nation; we must also recognise that among the nominal followers of an ideal there are always rival sections, doubtful adherents and secret rebels. Candid consideration of facts of this order suggests the doubt whether, strictly speaking, any ideal of life has more than one whole-hearted adherent: whether, if all men could know themselves and were sufficiently articulate, the number of ideals proclaimed to the world would not be exactly equal to the number of its human population!

This disturbing suspicion is greatly strengthened when we observe that the very conception of human beings as *individuals* implies the same conclusion. When we speak of a man as an individual we do not merely mean that he is an entity distinguished from other human entities by a particular name and address. What we have in view is something that makes him not only different from others, but unique. That something is his life—taking the word to cover all that issues from, or is expressed in, his physical, mental and spiritual nature. We need not puzzle ourselves with metaphysical riddles—asking whether two lives might not be so much alike as to make their owners (as the undergraduate said about S. John the Evangelist and S. John the Divine) "almost if not quite the same person." The common sense of mankind holds that different human beings are, through and through, themselves, and themselves only: that each contributes to the music of humanity a part which, however poor it may be, is yet his own and could never issue from another.

If this be true, it may be maintained, in a perfectly good sense of the words, that every man must have his own unique ideal. It would be the sense in which every work of art—for example, every poem—has its own ideal. A poet who recognises that his creative impulse has failed would never point to another poem and say, "that is what I intended to do." His ideal was concrete, and to be embodied, if at all, in *his* poem and in no other. It marks the perfect achievement from which *his* work has fallen short; not a goal that another has or might have reached.

From this train of thought it would follow that there can be no universal aim of education if that aim is to include the assertion of any particular ideal of life; for there are as many ideals as there are persons. Educational efforts, it would seem, must be limited to securing for every one the conditions under which individuality is most completely developed: that is, to enabling

him to make his original contribution to the variegated whole of human life as full and as truly characteristic as his nature permits; the form of the contribution being left to the individual as something which each must, by living and in living, forge out for himself.

Stated thus crudely this aim is not likely to win acceptance from every hearer. The impulse of some will be to dismiss it, regretfully, as impracticable; that of others to reject it, indignantly, as immoral. Since, however, it follows logically from a tenable view of life, let us waive for the moment the question of its practicability and see what can be said in defence of its moral character.

We may anticipate that the first charge to be met is that it permits no discrimination between good and bad ideals of life: between forms of individuality that ought to be encouraged and forms that ought to be suppressed. Is the schoolmaster, it will be asked, to foster with sympathetic impartiality the making both of an Émile Pasteur and of a Cesare Borgia?

The counsel for the defence must no doubt make a concession here. He will grant that the educator's duty does not require him to foster a bad life on the ground that it promises to achieve the uniqueness of a good poem. But he will point out that this admission does not invalidate the aim; it simply limits it to the field encircled by the moral law. Within that enclosure there remains infinite room for its application. A cook's life, for instance, may be as blameless as a missionary's; the world needs them both, and becomes richer as each becomes better after his own kind. Even where the moral law is positive, obedience takes forms endless and incalculable; "the spirit bloweth where it listeth." A motorist (it has been said) best shows his love for his neighbour by keeping to the left of the road. The point need not be laboured. It is manifest that there is no limit to the number of life-patterns into which good and blameless actions may be woven, and that it is impossible to formulate in advance the concrete principle of excellence of any of them.

The advocate may easily go further, and emphasise the difficulty of foreseeing whether a new type of individuality, a new form of expression in thought or action, will ultimately add to or detract from the real riches of the world. It is fatally easy to condemn as contrary to beauty, truth or goodness, what merely runs counter to our conservative prejudices. We know how often in the past men have sought to suppress the creative activity whose fruits have later been seen to be among mankind's greatest treasures. We need to remind ourselves—teachers, perhaps, more than laymen—that these "old, unhappy, far-off things" are constantly being repeated and paralleled; now in trivial instances, now in matters of serious importance. The younger of this generation have seen the "futurists" break away from the tradition of painting—a portent met here with ridicule, there with anger, with active reprobation all round. Their elders remember that the same phenomena attended (how incredible it is!) the emergence of Wagner and the "music of the future." And how long is it since the entrance of women into medical studies shook Victorian England to the very foundations of its respectability? *E pur si muovo.* The teacher will do well, then, always to have in mind the warning of Gamaliel * and to beware lest haply he should be found to fight against God.

But the accusation of immorality may be presented in a subtler form. The proposed aim, it may be said, heads straight for anarchy and "the breaking up of laws"; for it ignores the obligations of the individual towards society. Here we are brought face to face with a problem which, precisely because (from the days of Antigone downwards) it has given rise to some of the most perplexing and tragic situations in life, is necessarily a crux for educational theory. The counsel for the defence will, no doubt, avoid raising the more

* Also, perhaps, the aspiration of Anatole France: "Espérons dans ces êtres inconcevables qui sortiront un jour de l'homme, comme l'homme est sorti de la brute. Saluons ces génies futurs!"—*Le jardin d'Épicure.*

troublesome issues, but on the general question he may be expected to submit to the court some such plea as the following.

In the first place, the fact that social obligations are not explicitly mentioned in the aim does not mean that they are either ignored or overlooked. They cannot be ignored, for they are an essential part of the material out of which individuality is fashioned. The view that social bonds are imposed upon the individual from without and accepted under the terms of a "social contract" has long been exploded. It is now universally recognised that they originate within man's nature and are inextricably woven into the texture of his being. The strongest creative impulses cannot fashion a life which would not fall to pieces if the social elements were withdrawn. The most "original" personality is unintelligible apart from the social medium in which it grows; and no Thoreau could hide himself so deeply in the woods as to escape from the social in his own mind. We need not require, therefore, that our pupils shall be taught to feel the force of social bonds; the need is rather that budding individuality should not be warped from its ideal, and bent by influences "heavy as frost and deep almost as life."

Again, what was observed of positive morality holds good equally here. Social obligations can be discharged in an infinite number of ways, and none can foresee or set bounds to what the human spirit may do in this as in all other directions. A daring and powerful soul may raise the whole moral quality of the social structure by asserting an individuality that may at first seem hostile to its very existence. And the unheroic soul, too, will best serve society by becoming most fully and truly himself. In short, the claims of society upon its members are best satisfied not when each is made as like his fellows as possible, but when, in Dr. Bosanquet's language, "he values himself as the inheritor of the gifts and surroundings that are focussed in him, and which it is his business to raise to their highest power."

At this point the advocate may be supposed to turn from those who mistrusted the morality* to those who doubted the practicability of the aim he is defending. He will not fail to point out to the latter that individuality is not at all the same thing as eccentricity. The crowd and the hero have both such potent influence that few of us are likely in our development to wander far from the established types. But even though I choose, for example, to dress myself as nearly as possible like my companions, the fact that the choice is my own is the circumstance that should count. Moreover, I shall certainly wear the clothes with a difference which, though small, is all my own. In other words, my individuality consists not in the fact that I am different from others (though I am sure to be different), nor in any deliberate seeking after difference, but in the fact that what I am springs from my own nature (including the social tendencies that are part of it) and is not an artificial product of the will of another.

From the practical standpoint it appears, then, that the aim we are considering makes no extravagant demands upon educational administration; it does not require a separate school or a separate teacher for every pupil.

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* The acute reader will notice that our advocate has avoided by far the most difficult question: namely, whether society (or, to be precise, the state) may not at times of crisis demand from its members services that entail the supersession, even the final sacrifice, of their individual development, and whether an affirmative answer would not greatly weaken the general force of his argument. To this we may imagine him to reply that mankind is not condemned for ever to endure its present evils; if there is a will to escape from them, its nobler spirits ("Saluons ces génies futurs!") will certainly find a way. But if it is lawful to dream of a world in which the good of all would be much more nearly the good of each than it is at present, it is lawful to do whatever may help to make the dream reality. What, then, could education do better than to strengthen men's sense of the worth of individuality—their own and others',—teaching them to esteem the individual life, not, indeed, as a private possession, but as the only means by which real value can enter into the world? In this, it may be claimed, is the strongest bulwark of freedom and the firmest guarantee against the rule of violence.

What it does require is that school government and teaching shall be so conducted, and educational institutions so planned, that while the undistinguished child may be tempted and helped to throw himself with spirit into the business of making (or "realising") himself, the brighter and stronger souls, on whom the progress of the world mainly depends, shall not fail to find the nutriment they need in order to grow to the full height of their spiritual stature.

You will observe that I have delegated the expression of these views to a supposititious advocate. Is it to be understood, you will ask, that I have briefed this gentleman, or that I decline responsibility for his arguments? The question is a fair one, and I will answer it frankly.

In the first place, then, I admit that the counsel's position is one I find it impossible to repudiate. I become more firmly persuaded of its essential soundness the more I reflect on the problems of education and the more I study the attempts that have hitherto been made to solve them. In my judgment it is congruent with all the surer teachings of philosophy, science and history about the nature and destiny of human beings. It is, I think, manifest that our traditional school methods are largely based upon presuppositions and infected with ideas radically opposed to those teachings. They assume that it is the teacher's duty to play the part of a subsidiary Providence in children's lives and assign no limit to his activities in this capacity. To realise what this assumption implies, imagine what your attitude would be towards a person who should take in hand to "form" both your tastes and your character, and—here is the gravamen—refused you the right to reject his unsolicited ministrations. Can we maintain, without far-reaching qualifications, that what would be an outrage upon maturity is nevertheless a sacred duty towards youth? If we think so it is because we have far too little faith in the creative impulses that are the mainspring of every young life, and are, therefore, far too timid in exploiting them. Moreover it is to be feared that we have far too little reverence for the forms of individual existence which only those impulses can shape and maintain. I agree with my advocate that if we could bring ourselves to entrust to boys and girls a much greater share in their own education the results would fully justify our courage and our self-restraint.

On the other hand, though I firmly believe these things I can bring but little first-hand evidence to their support from my experience as a schoolmaster. My apprenticeship was served under the older tradition. Hence, although (as the ancient moralist said) I both see the better way and approve it, I am not confident that my engrained professional habits would permit me to follow it with success. It was, you will understand, a delicate sense of this weakness in my position that caused me to express my convictions by the lips of another!

There is, however, a rapidly growing company of witnesses to the practical efficacy of the ideas which my advocate defends. Justice as well as courtesy demands the place of honour for Dr. Maria Montessori. I am not to be understood as accepting her "system" as a whole, but no unbiassed observer can, I think, doubt that the influences of her doctrine and her personality promise to revivify the whole field of primary education. At the other end of the school period there is the wonderful work accomplished by Mr. Homer Lane in the Little Commonwealth. Among the numerous experiments inspired, directly or indirectly, by these great teachers one calls here for special mention, if only because it indicates that the new ideas have power to transform even that formidable thing, the public-school tradition. I refer to the experiment in self-government described with admirable candour and clearness by Mr. J. H. Simpson in his recently published book *An Adventure in Education*.

The significance of these essays in revolutionary pedagogy is chiefly in relation to school government and discipline. They show that the old authoritarian ideas on these matters, which we thought as firmly based in the nature of things as the everlasting hills, were, after all, only prejudices

or, at best, conventions. They prove that when those ideas are superseded, the school society, instead of dissolving in hideous ruin, may actually become a vastly healthier medium for young people to live in and to grow in. This discovery is of fundamental importance and must in time greatly change the character of our scholastic institutions. But the question to which I wish to direct your thoughts concerns the bearing of the new notions upon teaching method; for we must, I think, agree that it will be impossible to keep the new wine in the old bottles. Our present methods of class instruction belong integrally to the old tradition of school organisation and discipline. As that tradition disappears those methods must vanish also or become profoundly modified.

Here again many energetic reformers are and have long been at work. As might be expected, the most striking results hitherto obtained are in the teaching of the arts and crafts and music. Genuine culture in these subjects —so vitally important and yet so deplorably neglected—is impossible unless based upon the pupil's discovery and exercise of his own creative powers. That is why teachers who can exploit those powers have brought about what seems to others an almost incredible revelation of human faculty. Forgetful of the long-buried tradition of our greatness as a nation of artists, craftsmen and musicians, we think of the power to produce beauty as a gift granted only to the few. These teachers have rediscovered that it is almost as universal as the capacity to do arithmetic.

As regards the "intellectual" subjects of the curriculum, Professor Armstrong's famous propaganda undoubtedly opened the way for still more radical reformers. In mentioning these it would be ungrateful not to pay a tribute to the pioneer work of Mr. M'Munn. Mr. Caldwell Cook has shown in his vigorous and entertaining book, *The Play Way*, what may be done in the teaching of English by a courageous desertion of the didactic attitude and a bold encouragement of the pupil's spontaneity. Many enterprising men and women are making experiments along similar lines in other subjects. I instance a friend, headmaster of an ancient grammar school, who has entirely abandoned the class-method in teaching Latin and Greek, and has found his wisdom justified even by the examiners for classical scholarships. He forbids me at present to mention his name, but promises to make his experience public when it is more mature.

It will, perhaps, be thought that while other subjects may, without great harm resulting, be handed over to these subversive enthusiasts, mathematics must at any rate be preserved from them. If we allow a boy to roam where he pleases in mathematics, and if our rôle is merely to follow his lead, what will become of our curriculum, with its venerable hierarchy of ideas and processes and the nice gradation of difficulties based on the experience of generations of teachers? What, above all, of the logical training which many of us value as the chief function of mathematical teaching? I own that I am little moved by the fears these questions express. I have long held that in mathematics the "play way" is the best way, provided it can be followed under proper conditions and (I must add) with proper seriousness. I reached that view by studying the progress of a boy whom I once had excellent chances of observing. His official instruction in mathematics was very meagre; I regret to say that he attached to it no importance whatever. But he had the adventurous imagination which is the common and supreme treasure of boyhood, circumstances gave him rather unusual freedom to indulge it, and he had access to a reasonably liberal library. He developed a passion for surveying on the grand scale, for astronomy, for the lore of men who go down to the sea in ships. I remember that he once absented himself from a Local Examination—to be precise, he "cut" the Scripture paper—in order to observe that rare phenomenon, the transit of Venus. He reflected that while one may run away from an examination and live to pass it another day, a transit of Venus, once missed, is missed for ever. These tastes could

not be pursued without mathematics, so he acquired a store of shockingly illegitimate knowledge of that subject. He could use sines and cosines long before he knew the Sixth Book of Euclid and seven-figure logarithms before he had heard of the theory of indices. He employed complicated formulae of spherical trigonometry to find the hour of day by the sun without having the least notion how they were established. I must, however, do him the justice of saying that he was never contented with this empiricism, and was glad enough to know the why and wherefore of truths that seemed relevant to his favourite pursuits. In short, his self-acquired mathematics were scandalously scrappy and scandalously illogical, but had for him a reality and a substantial importance compared with which the discourses of his accredited instructors were as the crackling of thorns under a pot.

Nevertheless it is not a paradox to say that this boy's chief need was a teacher. But he should have been a teacher who conceived it his function to be, as Dr. Montessori is fond of saying, an observer. This does not mean a merely passive observer, contented to watch the human specimen and to enter his peculiarities in a note-book; but an active observer, prepared to "stand by" in the nautical sense of the term: that is, ready to lend a hand when a hand is needed, but not given to officious interference when interference is not called for. Under such auspices he might have become, in a small way, a real mathematician. As it is, he has to be contented to know that such mathematics as he learnt was not a dead burden, carried for a while and then dropped with relief; but is something that helps to direct and to colour his outlook on the world and is an essential vehicle of his self-expression: in a word, that it is woven into the fabric of his individuality.

I have given some time to my young acquaintance, not because he is exceptional, but, on the contrary, because he is typical. There are abler boys and there are duller; but, whether able or dull, all have their impulses that can be satisfied only by knowledge. I am convinced that the ideal way to teach all of them is, in principle, the same: namely, to put them where those impulses are likely to be awakened, and then to "stand by." In mathematics, boys' impulses generally follow a few well-recognised tracks, but one must be prepared for divergencies, sometimes unexpected, occasionally startling. I once had a pupil—I am now well content to be his—who at the age of fifteen was deep in the theory of equations. At his side sat a second who would look at nothing except the mathematics of electrical engineering. And there were others. "Standing by" was in those circumstances often a strenuous form of exercise, but I look back upon that period in my professional experience with less dissatisfaction than upon any other.

These small experiments of mine were conducted in a Sixth Form, where the atmosphere is serene and the bonds of class work and discipline may easily be, and commonly are, relaxed. Many of my audience have no doubt done the same sort of thing in similar circumstances. But what our reformers envisage is nothing less than the adoption of this type of teaching throughout the school. The moment such a prospect is presented certain difficulties inevitably arise in one's mind. Let us briefly consider the more important.

The basal principle of the new gospel is that primacy must be given to the pupil's bent; for the real driving power in education is always from within, never from without: it is a *vis a tergo* not a *vis a fronte*. What guarantee is there, then, that the pupil's bent will lead him to learn the things he ought to know? Is he to be allowed to leave school ignorant of essential studies simply because he happens never to have felt their attraction?

To this question the reformer might fairly reply with another. Does the disaster you contemplate never happen under the present system? Do you really succeed, by "forcible feeding," in making your pupils assimilate what you think they ought to know? But the *tu quoque* form of argument generally covers an attempt to evade an issue. Let us face the issue, and agree that if a child does not give himself spontaneously to essential studies

he must be made to do so. It must not be assumed too readily that the compulsion need be external. At the Caldecott Community, for example, where the possibilities of "individual education" are being explored with admirable scientific caution, children of seven are taught to plan each day's work at the beginning of the morning, while older children map out in the same way their proposed activities for each week. As a matter of course they submit their programmes to their teachers. If essential but unattractive subjects seem to have less than their due, the defects are pointed out. The better judgment of the child as a rule accepts the criticism and makes it his own.

This is a simple instance of a procedure that must often be adopted if the new method is to be used with discretion and not followed with the blindness of the doctrinaire. External compulsion, except as a remote and last resort, has no place in the scheme; but persuasion and the prestige of age and experience retain their natural right to influence conduct. Indeed their sway is all the more powerful when there is no suspicion that they are merely the kid glove hiding the mailed fist. Thus, even Mr. Bernard Shaw, who allows to children the extreme of liberty in self-determination, still claims for the teacher the right to "bias" the child in the exercise of his freedom.*

In the next place, it must be admitted that the number of things that absolutely must be known by an educated person is rather surprisingly limited. When one has set down the power to read with ease and comprehension, the power to express oneself fluently and correctly in speech and writing upon subjects one understands, the power to do simple computations, the list is almost if not quite exhausted. Beyond these things there is much it is desirable to know; much, if you please to put it so, that everyone ought to know; but little that everyone must know. It is, I agree, deplorable that a boy or girl should leave school with wisdom at one of her main entrances quite shut out; but in saying so I am voicing, I suspect, the prejudice of a schoolmaster rather than the judgment of the world. Even the little world of "cultivated" people is by no means constant in its demands. Herbert Spencer complained bitterly that the man who would be covered with confusion if he misplaced the quantities in the word Iphigeneia could admit, unabashed, complete ignorance about his Eustachian tubes. Before very long the conventions of polite society may be reversed: for instance, a knowledge of Mendelism may become the "freemasonry of gentlemen" and a reference to Horace be regarded as a lapse from good taste. Meanwhile the greater world is widely tolerant of ignorance in most matters provided it is outbalanced by competent knowledge in others. We might with advantage, perhaps, admit more of this tolerance into the school.

But special cases of the kind now contemplated are and always will be rare. The great majority of boys and girls are like the busybody in Terence's play—nothing that possesses permanent and substantial interest for humanity leaves them unconcerned. Here, expressed in brief, is the psychological sanction for the school curriculum. If a subject that represents one of the main historical lines of human interest fails to attract a healthy young mind at the normal age the fault is most probably in the presentation. The teacher should try again.

This argument brings us to the second of our problems. If the teacher is not to teach nor, it would seem, to "keep order," what functions are left to him? As regards "keeping order" I must, for brevity's sake, refer the inquirer to Mr. Homer Lane and Mr. Simpson. As regards instruction, it is certain that the teacher's functions will not be less important or less difficult than at present. He must, as we have seen, practise certain negative virtues:

* Mr. Shaw has devoted one of his formidable "prefaces" to an acute and eloquent discussion of the main topic of this address: "Parents and Children," in the volume of plays entitled *Misalliance*.

he must keep the *docendi cacoethes* under firm restraint and learn to "stand by." But in addition he will have positive responsibilities whose exercise will give full scope to professional cunning. It will be his business to create an environment in which his pupils' impulses towards mathematical knowledge will naturally be wakened, and to shepherd them unobtrusively in the right directions. Himself steeped in the best traditions of his subject, it will be his task to see that by suggestion, inspiration and criticism those traditions enter into and form the minds of his pupils. He will be an "idea-carrier" between the great world and the artificial microcosm of school, infecting his pupils imperceptibly with germs that will fructify into ideals of sound workmanship and devoted labour.

Lastly, there is the difficulty of school organisation. It is incontestable that the rigid class-system and the rigid time-table are incompatible with the new method. For my own part I should rejoice to see those institutions replaced by others less violently unnatural. You all know how a familiar word, persistently stared at, suddenly becomes almost alarmingly strange and meaningless—how it seems to glare back from the page with no speculation in its eyes. You will have something like the same uncanny experience if you watch the operation of a school time-table after rigorously clearing your mind of its familiar associations. From 10.15 to 11 twenty-five souls are simultaneously engrossed in the theory of quadratic equations; at the very stroke of the hour their interest in this subject suddenly expires, and they all demand exercise in French phonetics! Like the agreement of actors on the stage, "their unanimity is wonderful"—but also, when one comes to think of it, ludicrously artificial. Can we devise no way of conducting our business that would bring it into better accord with the natural ebb and flow of interest and activity? It may be that the specialist system, often a tiresome complication of the present arrangements, would make a more fluid organisation perfectly feasible. There must still be, no doubt, certain fixed periods for collective work; but during the rest of the day each specialist's room might be a "pupil room" in which boys or girls of all standing would work, singly or in small groups, in independence of one another and for variable lengths of time. It would, of course, be necessary to record each pupil's progress and to see that he followed a reasonable programme of studies; but I find no reason why in such matters methods like those of the Caldecott Community should not be universalised.

But it is time to check the flight of speculations which have already travelled, perhaps, too far from actuality. Indeed you may think that, like the poet, I "sing of what the world will be when the years have died away," but of what it can never be in finite time. That is quite possible; progress towards an ideal is necessarily asymptotic. The important question is whether the ideal marks out a desirable line of movement. I have confessed that freedom for each to conduct life's adventure in his own way and to make the best he can of it is for me the only universal ideal that has the support of nature and the approval of reason; the beckoning gleams of other ideals are but broken lights from this. That freedom is the source of all the higher goods. Apart from it duty has no meaning, self-sacrifice no value, authority no sanction. It offers the one possible foundation for a brotherhood of nations, the only basis upon which men can join together to build the city of God. Hunger for it is the secret source of much of the restless fever of our age. By a paradox as superb as cruel, millions of men who share our speech and millions more who share our hopes have given up their own claims to it so that in the end it may become the law of the world. For if "to make the world safe for democracy" means not this, it can mean nothing but to exchange one tyranny for another. Dare we take a lower, and can we find a higher, ideal to be our inspiration and guide in education?

T. P. Nunn.

## ADVANCED SECTION.

# THE GRAPHICAL TREATMENT OF POWER-SERIES.

By William P. Milne, M.A., D.Sc.

The object of the "Advanced Section" being to study and find out the best methods of teaching Scholarship work, it seems advisable briefly to pass in review the higher algebra syllabus which is at present being taught to pupils in the departments of advanced mathematics in our secondary schools.

The first and most vital criticism is that higher algebra and the calculus continue even at the present day to be taught in water-tight compartments as separate subjects, whereas modern investigations show conclusively that the proper line of advance consists in applying the methods of algebra-cum-calculus. The second criticism is that the whole tone of our higher algebra schedule has little to recommend it except that it is hallowed by tradition. Its outlook is thoroughly archaic. It is a patchwork affair with no coherent pattern running through and connecting the constituent elements. The various chapters depend little upon one another and are very slenderly interrelated. Let us examine details. The treatment of the theory of numbers is perhaps the most satisfactory, but there are too many formulae and too little direct handling of concrete numbers. Continued fractions have many useful and easy applications, for example, to differential equations. Of these applications the student gets not a glimpse. Instead he gets a huge abstract theory and learns to do tricks with $p_n$ and $\gamma_n$, but of what practical use the engine is, he is left utterly in the dark. The beautiful applications of determinants to invariant-geometry find no place in our scheme. Determinants are determinants and no more. The solution of equations and power-series by numerical methods and the application to practical problems receive only the scantiest attention. Gräffe's numerical solution of equations and power-series by dispersion of the roots is unknown. On the theoretical side, of the interrelations of the cubic and quartic equations with the geometry of conics and triangles not a glimpse reaches our pupils. Just think how in dealing with the quartic equation we present Euler's Reducing-Cubic as a chance stray-shot amidst a mass of unilluminating algebraic manipulation, and contrast with what irresistible vividness and force Euler's equation presents itself as the **three** degenerate line-pairs of the pencil of conics through **four** points. I have left to the last the greatest blot of all. We study tricks with **three** series and **three** only, namely the binomial, exponential and logarithmic; and we give elaborate and rigorous proofs invented specially for each, which only the best boys can understand, and few even of the best can reproduce and really assimilate. Other power-series we do not teach. General ideas on power-series are utterly ignored. As aforesaid, modern mathematical advance is taking place to a large extent not by algebra only and not by calculus only, but by the two combined, with the power-series as the fundamental weapon, and it is this aspect of the study of mathematics for Scholarship boys that I intend to put before you to-day.

I base the presentation of power-series as far as possible on the methods of decimals, so as to utilise unconsciously the boy's powers of generalisation and intuition, and I therefore begin with "contracted methods."

## I.

*To multiply* $2+3x+7x^2-5x^3-2x^4+9x^5$ *by*

$$3-4x+5x^2-7x^3-11x^4+8x^5-4x^6$$

*correct to the third power of* $x$.

We ignore all powers of $x$ beyond the third and proceed as shown below,

scoring off one term in the multiplicand after multiplying by the successive powers of $x$ in the multiplier.

$$\begin{array}{r} 2+3x+\ 7x^2-\ 5x^3 \\ \hline 3-4x+5x^2-7x^3 \\ \hline 6+9x+21x^2-15x^3 \\ -8x-12x^2-28x^3 \\ 10x^2+15x^3 \\ -14x^3 \\ \hline 6+x+19x^2-42x^3 \end{array}$$

## II.

*To divide* $7-4x-5x^2+11x^3-3x^4+10x^5$ *by*

$$1-2x+3x^2-4x^3+11x^4-17x^5+10x^6$$

*correct to the third power of* $x$.

We ignore all powers of $x$ beyond the third and proceed as shown below, scoring off one term in the divisor after each division.

$$\begin{array}{r} 1-2x+3x^2-4x^3)\,7-\ 4x-\ 5x^2+11x^3\,(\,7+10x-6x^2-3x^3 \\ 7-14x+21x^2-28x^3 \\ \hline 10x-26x^2+39x^3 \\ 10x-20x^2+30x^3 \\ \hline -6x^2+\ 9x^3 \\ -6x^2+12x^3 \\ \hline -3x^3 \\ -3x^3 \\ \hline \end{array}$$

## III.

### Approximating-Curves.

The next thing is to show that a vast number of complicated functions of $x$ can be replaced to any assigned degree of approximation by polynomials, and this principle can best be illustrated and driven home by the copious use of squared paper.

*Example* 1. Consider $y=\dfrac{1}{x-3}$.

Divide by contracted methods as above, obtaining:

$$-\frac{1}{3}-\frac{x}{9}-\frac{x^2}{27}-\frac{x^3}{81}.$$

Consider then the following series of approximating-curves, which we expect intuitively to approximate indefinitely close to $y=\dfrac{1}{x-3}$ (at least for some values of $x$).

$$(1)\ y=-\frac{1}{3}-\frac{x}{9};$$

$$(2)\ y=-\frac{1}{3}-\frac{x}{9}-\frac{x^2}{27};$$

$$(3)\ y=-\frac{1}{3}-\frac{x}{9}-\frac{x^2}{27}-\frac{x^3}{81},$$

etc.

We graph the curve itself and the above series of approximating-curves, obtaining the following diagram:

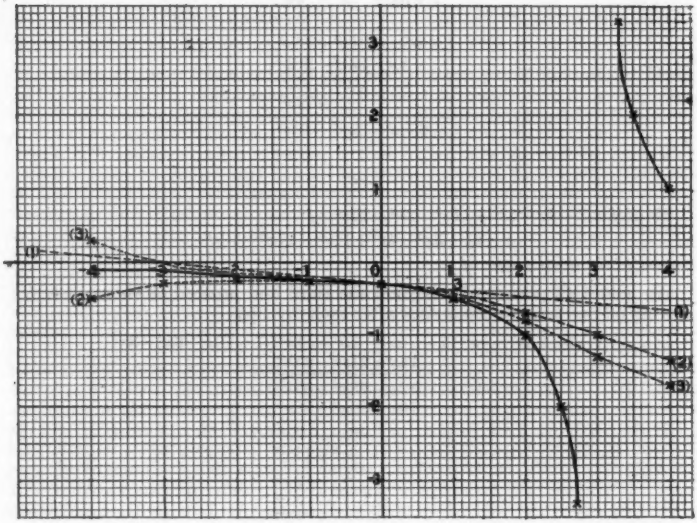


FIG. 1.

Certain facts are to be noticed at once:

(1) Only for some distance on both sides of the $y$-axis do the approximating-curves tend towards the given curve $y = \dfrac{1}{x-3}$ as a limit.

(2) On the right-hand side of the $y$-axis, where all the terms in $x$ appearing in the various equations are *of one sign*, namely negative, the approximating-curves bend steadily down, and each one is a closer approximation to the given curve than the one before it.

(3) On the left-hand side of the axis of $y$, where the terms in $x$ appearing in the various equations are alternately negative and positive, the approximating-curves are alternately above and below the curve to which they tend as the limit, and each one is a closer approximation to the given curve than the one before it.

(4) The original curve and the approximating-curves all touch one another at the point where they cut the $y$-axis; and the closeness of the contact between the given curve and any approximating-curve is greater if the approximating-curve be a later than if it be an earlier one.

(5) It is plain from the diagram that between $x = -3$ and $x = +3$ the approximating-curves *seem* to tend to the given curve $y = \dfrac{1}{x-3}$ as the limit, but that to the right of $x = 3$, the approximating-curves most certainly do not tend towards the original curve as the limit. Towards the left of $x = -3$, the approximating-curves seem to diverge further and further from the assigned curve, first above and then below. This phenomenon will be called later on the "Dying-Fish Phenomenon." It is an important and character-

istic one, and takes its name from the violent oscillations of the tail in the fish's dying agony.

These phenomena, which the squared paper leads us to expect, can be proved very easily by algebra. We shall call the interval $x=-3$ to $x=+3$, between which (exclusive of the end-values) the approximating-curves tend towards the given curve as the limit, "**the range of validity**" of the approximation.

*Example* 2. Consider $y=(1-x)^{\frac{1}{2}}$.

Let us extract the square root by the usual elementary method, and we have

$$y=1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}+\text{etc}\ .$$

We obtain the following series of approximating-curves and graph them:

$$(1)\ y=1-\frac{x}{2};$$

$$(2)\ y=1-\frac{x}{2}-\frac{x^2}{8};$$

$$(3)\ y=1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16};$$

etc.

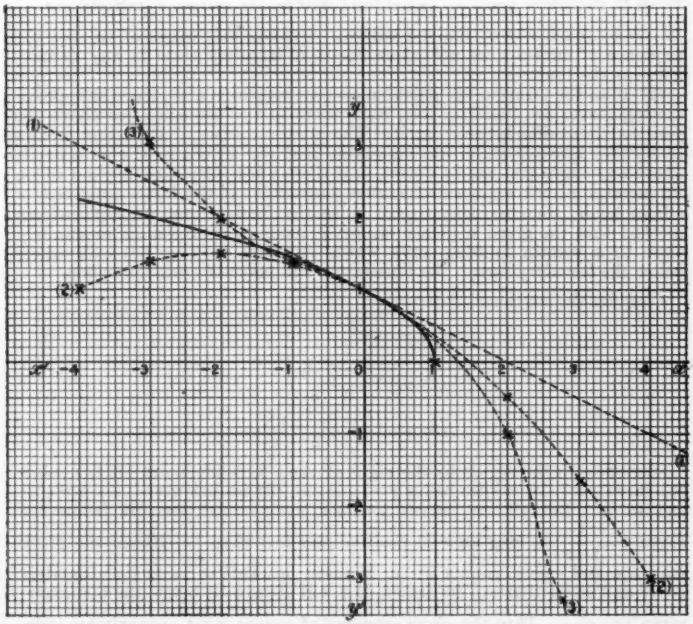


FIG. 2.

Figure 2 brings out very clearly the existence of a "*range of validity*," inasmuch as the assigned curve $y=(1-x)^{\frac{1}{2}}$ stops dead at $x=1$, whereas the approximating-curves continue their existence to the right of $x=1$. To the left of $x=-1$ one sees the "Dying-Fish Phenomenon," viz. the oscillations of its tail above and below the assigned curve as it writhes in its death-throes, the oscillations becoming more and more violent as the later approximating-curves are taken.

## IV.

### Ranges of Validity and Boundary-Curves.

There is no general rule for finding the range of validity for a given system of approximating-curves, but we shall indicate one or two simple methods of procedure.

(1) Where the sum of the terms of the equation to each approximating-curve can be expressed in a simple form, it is easy to find the range of validity and the curve to which they approximate. The ordinary geometrical series is the simplest example of this. Thus the sequence of approximating-curves

$$(1)\ y=1+x;$$
$$(2)\ y=1+x+x^2;$$
$$(3)\ y=1+x+x^2+x^3,$$
$$\text{etc.}$$

tends to the curve $y=\dfrac{1}{1-x}$ as the limit within the range of validity $x=-1$ to $x=+1$ (exclusive of the end-values). Outside this range of validity the sequence of curves does not approximate indefinitely close to any definite curve as the limit.

(2) If all the terms in the polynomials that represent a sequence of approximating-curves are positive, then each curve in the sequence lies above the one that immediately precedes it. The question arises, "Does this process of *bending upwards* go on indefinitely so that the sequence of curves bends above any curve that can be drawn, or is there a *boundary-curve* beyond which no member of the sequence of curves passes?" If the latter condition of affairs holds, then we may regard it as the fundamental principle of our method that the given sequence of curves tends to approximate indefinitely close to a certain limiting-curve which coincides with or lies everywhere below the *boundary-curve*. An illustration will make the principle plain.

Consider the sequence of curves:

$$(1)\ y=1+\frac{x}{2};$$
$$(2)\ y=1+\frac{x}{2}+\frac{x^2}{3};$$
$$(3)\ y=1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}.$$

Obviously in the region to the right of the $y$-axis the curve (1) lies below the curve $y=1+x$, (2) lies below $y=1+x+x^2$, (3) lies below $y=1+x+x^2+x^3$, and so on. But all the curves $y=1+x$, $y=1+x+x^2$, etc., lie below the curve $y=\dfrac{1}{1-x}$ so long as $x$ is positive and less than 1. Hence the given sequence and curves cannot go on bending upwards so as to lie ultimately above any curve that can be drawn, but will always be below the boundary-curve $y=\dfrac{1}{1-x}$. Thus the given sequence of curves must approximate inde-

finitely close to some ascertainable curve that either coincides with the above boundary-curve or lies below it throughout the region lying between 0 and 1.

Next consider the sequence of curves:

$$(1)\ y=1+2x;$$
$$(2)\ y=1+2x+3x^2;$$
$$(3)\ y=1+2x+3x^2+4x^3.$$

It is more difficult to see by inspection a boundary-curve for the above sequence of curves. We use a method devised by Gauss.

Consider the $n^{\text{th}}$ approximating-curve,

$$y=1+2x+3x^2+4x^3+\ldots+(n+1)x^n,$$

and let us write it in the form

$$y=1+2x+3x^2\left\{1+\frac{4}{3}x+\frac{5}{3}x^2+\ldots+\frac{n+1}{3}x^{n-2}\right\},$$

*i.e.*
$$y=1+2x+3x^2\left\{1+\frac{4x}{3}+\frac{4x}{3}\cdot\frac{5x}{4}+\ldots+\frac{4x}{3}\cdot\frac{5x}{4}\cdot\ldots+\frac{(n+1)x}{n}\right\}.$$

Now $\frac{4}{3}>\frac{5}{4}>\frac{6}{5}>\frac{7}{6}>\ldots$, etc.

Hence the $n^{\text{th}}$ approximating-curve lies below the curve

$$y=1+2x+3x^2\left\{1+\frac{4x}{3}+\left(\frac{4x}{3}\right)^2+\ldots+\left(\frac{4x}{3}\right)^{n-2}\right\},$$

*i.e.* below the curve
$$y=1+2x+\frac{3x^2}{1-\frac{4x}{3}},$$

throughout the range $x=0$ to $x=\frac{3}{4}$.

We have thus shown that the given sequence of approximating-curves does not bend indefinitely far upwards between the limits $x=0$ to $x=\frac{3}{4}$, but has a boundary-curve. Hence the given sequence of curves tends to approach indefinitely close to a definite ascertainable curve throughout the range of validity $x=0$ to $x=\frac{3}{4}$.

*N.B.*—Further investigation would show that the range of validity is really longer than $x=0$ to $x=\frac{3}{4}$, but we leave this for the present.

## V.

### Integration.

The following result is graphically intuitive:

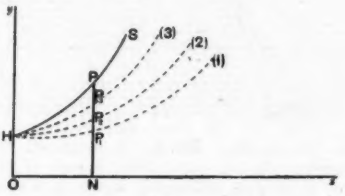


Fig. 3.

*In the above diagram, if* (1), (2), (3), ... *be a series of curves, approximating indefinitely close to the curve S, the areas* $ONP_1H$, $ONP_2H$, $ONP_3H$, ... *approximate indefinitely close to the area* $ONPH$ (*where in each case the curve acts as part of the boundary of the area*).

Thus we know that the following is a sequence of curves approximating indefinitely close to the curve $y=\frac{1}{1+x}$ throughout the range of validity $-1$ to $+1$:

(1) $y=1-x$;

(2) $y=1-x+x^2$;

(3) $y=1-x+x^2-x^3$.

Hence, taking the area between $x=0$, $x=X$, the axis of $x$ and the several approximating-curves, we obtain as the respective areas:

(1) $X-\frac{X^2}{2}$;

(2) $X-\frac{X^2}{2}+\frac{X^3}{3}$;

(3) $X-\frac{X^2}{2}+\frac{X^3}{3}-\frac{X^4}{4}$.

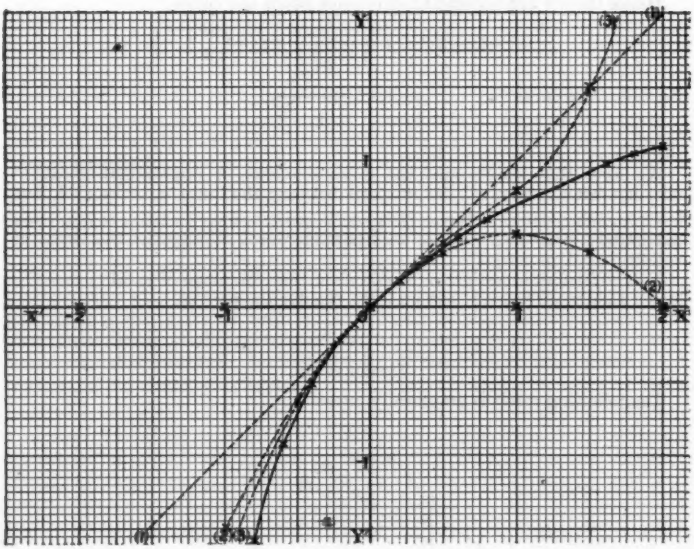


FIG. 4.

Finally, the area contained between $x=0$, $x=X$, the axis of $x$ and the curve itself $y=\frac{1}{1+x}$, is $\log_e(1+X)$.

Hence, by the above intuitive theorem of integration, we see that the sequence of polynomials (1), (2), (3), ... approximate indefinitely close to $\log_e(1+X)$, where $X$ lies within the range $-1$ to $+1$.

We may exhibit this result on squared paper for the sequence of curves,

$$(1)\quad Y = X - \frac{X^2}{2};$$

$$(2)\quad Y = X - \frac{X^2}{2} + \frac{X^3}{3};$$

$$(3)\quad Y = X - \frac{X^2}{2} + \frac{X^3}{3} - \frac{X^4}{4},$$

etc.,

where $Y = \log_e(1+X)$ is the curve to which they approximate. We note also, the characteristic properties to the right of the $Y$-axis, where the terms of the equations to the various curves are alternately + and −, viz. that the approximating-curves lie alternately above and below the limiting-curve, whereas to the left of the $Y$-axis, where the terms are all negative, the approximating-curves bend down towards the limiting-curve.

## VI.

### Differentiation.

The fundamental principle underlying the differentiation of approximating polynomials is equally simple from intuitive considerations:

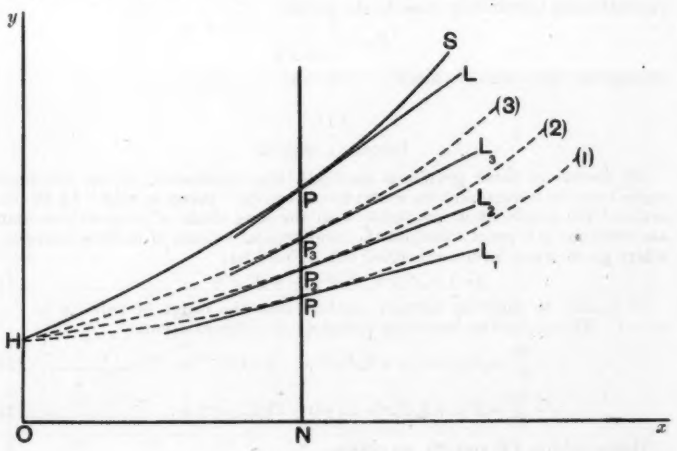


Fig. 5.

*If a sequence of approximating-curves* (1), (2), (3), ..., *tending to approach indefinitely close to the curve S, cut the ordinate ON at* $P_1$, $P_2$, $P_3$, ... $P$ *respectively, then it is obvious intuitively that the sequence of tangents* $P_1L_1$, $P_2L_2$, $P_3L_3$, ... *tends to approach indefinitely close to the tangent* $PL$.

Thus consider as before the sequence of curves,

$$(1)\quad y = 1 - x,$$

$$(2)\quad y = 1 - x + x^2,$$

$$(3)\quad y = 1 - x + x^2 - x^3,$$

etc.,

approximating to the curve $y=\frac{1}{1+x}$, the range of validity being $-1$ to $+1$. Consider the sequence of tangents at the points where the various curves are cut by the ordinate $x=X$. Their gradients are respectively :

$$-1,$$
$$-1+2X,$$
$$-1+2X-3X^2,$$
$$-1+2X-3X^2+4X^3,$$
$$\text{etc.},$$
$$-\frac{1}{(1+X)^2}.$$

Hence, by the above intuitive theorem of differentiation, we see that the above sequence of polynomials approximates to the expression $-\frac{1}{(1+X^2)^2}$ throughout the range of validity $-1$ to $+1$. Otherwise, viewed graphically, the sequence of curves,

$$(1)\ Y=-1+2X,$$
$$(2)\ Y=-1+2X-3X^2,$$
$$(3)\ Y=-1+2X-3X^2+4X^3,$$
$$\text{etc.},$$

approximates indefinitely close to the curve

$$Y=-\frac{1}{(1+X)^2}$$

throughout the range of validity $-1$ to $+1$.

## VII.

### Binomial Series.

By means of those graphical methods, the summation of the binomial series loses its terrors, and the result has not to be "taken as read" by 80 per cent. of the pupils, as is inevitable with the long chain of propositions that are essential to a proof based on the fundamental axioms of modern analysis, where geometrical intuition is ruled out. Consider :

$$y=1+{}_nC_1x+{}_nC_2x^2+\ldots+{}_nC_rx^r+\ldots. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

It is easy to show by Gauss's method that the range of validity is $-1$ to $+1$. Hence, by the foregoing principle of differentiation,

$$\frac{dy}{dx}={}_nC_1+2{}_nC_2x+3{}_nC_3x^2+\ldots+r{}_nC_rx^{r-1}+\ldots\ ; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\therefore\ x\frac{dy}{dx}={}_nC_1x+2{}_nC_2x^2+\ldots+(r-1){}_nC_{r-1}x^{r-1}+\ldots. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Hence, adding (2) and (3), we obtain

$$(1+x)\frac{dy}{dx}=n(1+{}_nC_1x+{}_nC_2x^2+\ldots+{}_nC_{r-1}x^{r-1}+\ldots)\ ;$$

$$\therefore\ (1+x)\frac{dy}{dx}=ny\ ;$$

$$\therefore\ \frac{dy}{y}=\frac{n\,dx}{1+x}\ ;$$

$$\therefore\ \log y=n\log(1+x)+C.$$

But when $x=0$, $y=1$ by (1), $\quad\therefore\ C=0\ ;$

$$\therefore\ y=(1+x)^n.$$

## VIII.

### The MacLaurin-Taylor Method of Interpolation.

Of those functions that present themselves in the elementary regions of mathematics, we can usually find direct from the definition the values of the function corresponding to several particular values of the independent variable. Thus we can easily calculate $\sin x$ when $x=0°$. $30°$, $45°$, $60°$, $90°$, and from these obtain in a general way the character of the graph $y=\sin x$. Similarly with $y=\log_{10}x$, when $x=10°$. Having obtained thus the general character of the graph by using several specified points thereon, we can calculate the values of the function corresponding to intermediate values of the independent variable by the method of interpolation devised by Taylor and Maclaurin. We note that in general the tangent at any point of a graph can be utilised to represent with very fair accuracy the value of the function at points in the neighbourhood of the point of contact. For example, consider the graph of $y=\sin x$, and let us find the tangent at the point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$. It is plainly of the form

$$\left(y-\frac{1}{2}\right)=m\left(x-\frac{\pi}{8}\right),$$

where $m\equiv\frac{dy}{dx}=\frac{\sqrt{3}}{2}$ at the point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$, giving

$$\left(y-\frac{1}{2}\right)=\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right).$$

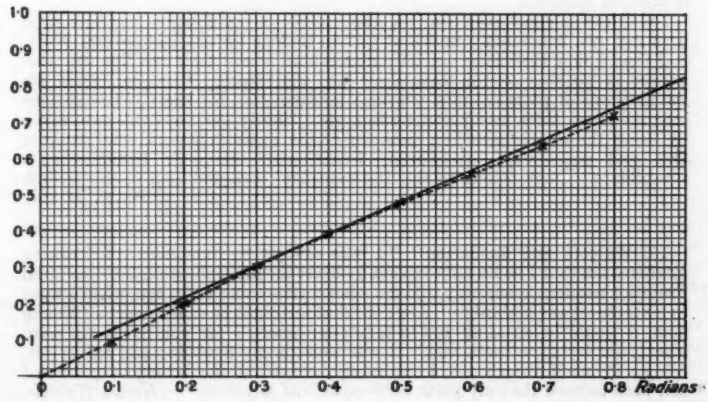


Fig. 6.

The graph shows that the tangent lies closely alongside the curve near the point of contact for a very considerable distance.

Suppose now that we wish to interpolate and to find the sine of 31°. We put $x=\frac{\pi}{6}+\frac{\pi}{180}$ in radians, *i.e.* from the foregoing equation

$$y=\frac{1}{2}+\frac{\sqrt{3}}{2}\times\frac{\pi}{180}$$
$$=0{\cdot}5151.$$

If we consult our tables for the sine of 31°, we get 0·5150. Hence the tangent represents the curve to a very considerable degree of accuracy in the neighbourhood of $x=\frac{\pi}{6}$.

The tangent must be considered as the first approximating-curve. Suppose we take as the second approximating-curve a curve of the form

$$\left(y-\frac{1}{2}\right)=\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)+p\left(x-\frac{\pi}{6}\right)^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Differentiating twice we obtain

$$\frac{d^2y}{dx^2}=2p. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

But the value of $\frac{d^2y}{dx^2}$ at the point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$ is the same for the approximating-curve (1) and for the original curve. Hence $-\frac{1}{2}=2p$, *i.e.* $p=-\frac{1}{4}$. Thus the equation to the second approximating-curve at the point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$ is

$$\left(y-\frac{1}{2}\right)=\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)-\frac{1}{4}\left(x-\frac{\pi}{6}\right)^2.$$

If we use this second approximating-curve to calculate sin 31°, we get 0·515039, which is only "one out" in the sixth decimal place, viz. 0·515038.

So far we have said nothing about the methods of estimating the errors involved in using approximating-curves instead of the curves themselves. That forms the second chapter in this method of presenting a First-Course in Modern Analysis.

I may say in conclusion that the above mode of presentation has been used with my own Scholarship pupils at Clifton College for the last several years, and that I find that they not only understand where they are, but feel confidence in pushing on for themselves into the future. Moreover, these are the methods of practical men who use mathematics for their investigations into the stars, tides, aeroplane motion, etc. Surely our boys will gain much by using the elastic intuitive methods of the great explorers, who conquer first and consolidate afterwards. Intuitionists conquer: Epsilonologists consolidate. Boys belong to the first category; bearded men to the second.

W. P. MILNE.

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**15. Walpole and Sanderson.** "I was always so incapable of learning mathematics that I could not even get by heart the multiplication table, as blind Professor Sanderson honestly told me, above three-score years ago, when I went to his lectures at Cambridge. After the first fortnight he said to me:—'Young man, it would be cheating you to take your money; for you never can learn what I am trying to teach you.' I was exceedingly mortified and cried; for, being a Prime Minister's son, I had firmly believed all the flattery with which I had been assured that my parts were capable of anything."—(*Horace Walpole*, Letters, ix. 467.)

**16. Solving Quadratics.** "When $b^2>4ac$ the expression $ax^2+bx+c$ has two real and differing roots [note the old use of 'roots'], contained in the formula *

$$-(b\pm\sqrt{b^2-4ac})/(2a).$$"

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* This formula should be committed to memory, and quadratic equations always solved by it. Nothing is more amusing than the vitality of the old method of completing the square and extracting the root in every particular case. No doubt a student should have some training in the last-mentioned process; but his ultimate method should be that of remembering, once for all, the formula in the text.

De Morgan [*Theory of Equations*, P.C. 1842].

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## GENERAL SECTION.

# THE USES AND FUNCTIONS OF A SCHOOL MATHEMATICAL LIBRARY.

By William P. Milne, M.A., D.Sc.

There is nothing more extraordinary in the educational world at the present day than the change of attitude among teachers towards the subject they have to teach. Teachers of bygone ages were to a large extent content to regard their pupils as so many buckets into which they were required to pour a prescribed amount of intellectual material. In most cases the teacher did not know himself the sources whence came his stock-in-trade. He had so many hard facts at his disposal which he had either got out of a printed text-book or from lecture notes, and it was his business according to his genius and ability to breathe upon these dry bones that they should live. It is possible for any of us to read the oft-quoted dictum of such a progressive and gifted teacher as Todhunter himself in one of his essays, wherein he brands any boy as wanting in the power of appreciating evidence if he does not believe the statements of his tutor—"probably a clergyman of mature knowledge, recognised ability and blameless character." Compare with that the attitude of teachers of the present day, which is consciously or unconsciously, explicitly or implicitly, utterly different. For the teachers of to-day, their subject is not a finished structure like an old mausoleum, beautiful outside but a thing of death within. It is an organic growth, ever growing, ever changing, ever being added to, ever having new methods devised and old cast aside. The old-fashioned attitude is typified by a most successful schoolmaster of the last generation who said to me on the eve of his retirement: "But if a man's a Senior Wrangler, what more is there for him to learn?" What an uncomfortable feeling a Senior Wrangler must have? I do not think that we quite realise what we owe to the Technical Colleges in getting rid of the old view. For the students of Technical Colleges, mathematics is essentially an instrument. They are willing to take much upon trust and to rely to a large extent upon that over-blessed word—*Intuition*. Furthermore, they are always on the outlook to use what they have learned, as a weapon. The result is inevitable. Mathematics acquired under these circumstances may be imperfect, may be rough-and-ready, but the subject is at all events alive; it is confined within no rigid barriers, and is eternally moving and growing and changing. In the Technical College class-room, the propositions of geometry do not pass before the audience in stiff and stately procession as the actors on the Greek stage: they are rather as the people of the market-place, hot and throbbing with life. Another great and incalculable influence in helping to overthrow the old rigid and detailed view of a subject lies in the fact that many universities nowadays insist that the student shall do some original work for himself in qualifying for an honours degree. Such research may not be of great scientific value in itself, but a bit of research stubbornly done and successfully achieved is part of a man's soul, and alters his whole attitude towards his subject. Men trained on research lines are slowly percolating through the staffs of the schools, and their attitude as teachers reacts on the taught. The combined result of all these movements and tendencies is that both master and pupil recognise that it is possible to know only a very little about this ever-growing ever-changing subject, but what knowledge they do possess fills them with an Oliver-Twist-like eagerness to ask for more. There is in fact a wide-spread feeling among teachers of mathematics that every school ought to possess an up-to-date mathematical library, so that both teachers and taught may

keep abreast of the times in their school work and also catch a glimpse of the great regions of mathematical thought that lie ahead. The present paper discusses this view and deals in a general way with the constitution of such a library.

The school library should consist of two parts—one for scholars and one for masters—though these two should not be mutually exclusive. In general tone and constitution the masters' portion should be altogether heavier and more serious than the scholars', because the teachers are older, more experienced and have greater width of knowledge and intellectual power than the scholars. On the other hand, the senior division of the library is not intended for highly-specialised experts in various branches of mathematics. To provide literature and reference books for these is the function of a university library. The books of the teachers' library should present rather "First Courses"—as they are popularly called—in the various subjects in which they deal. What schoolmaster does not want to know something about the great modern subject of Statistics with its manifold practical applications? Who of those present to-day will not want to know something more about that rapidly extending subject of Nomography, concerning which Dr. Brodetsky is to speak? What modern teacher of mathematics does not want to know something about the modern mathematical theory of gunnery, submarines, stability of aeroplanes—the last of which has been dealt with in such a fascinating manner by Prof. G. H. Bryan, of Bangor? On the other hand the scholars' library should be more suited to their age and capacity: mathematical recreations and puzzles; Rouse-Ball's ever-fresh *History of Mathematics*; elementary history and theory of Astronomy; the recent book on *Discovery* by Prof. R. A. Gregory; *Pioneers in Science* by Sir Oliver Lodge, and so forth. The boy must be able to get his hands on such books easily and be encouraged to read them.

One cannot and ought not to lay down hard and fast rules about the composition of the school library. It depends so largely on the individual tastes of the members of the mathematical staff and the funds at their disposal. A convenient method, however, would be to collect and catalogue the books under the headings of: (1) Biography, (2) History, (3) Philosophy, (4) Mathematical Analysis, (5) Geometry, (6) Applied Mathematics, (7) General. Such a composition gives plenty of scope for variety and elasticity. Most of these departments and subjects speak for themselves, and there is no need for me, therefore, to deal with them in detail, but I do want to say a word about one branch of mathematical knowledge, the total neglect of which in our curricula I have very much at heart. I refer to the department of History and Biography. The chemists do not neglect the history of their science, as witness the essays of Sir William Ramsay, Tilden, Shenstone, and a score of others. The musicians are not unmindful of the historical development of their subject. All well-educated musicians know something about the life, character, and historical position of the Bachs, Mozarts, Wagners of their craft. Think of that most fascinating volume, Sir Hubert Parry's *Lives of the Musicians*. Is a teacher of English literature conceivable who does not know that Shakespeare was born at Stratford-on-Avon? Think of the generality of mathematicians and what they know about the heroes of their science! I draw a sorrowful veil. There are two sides to the acquisition of knowledge on any subject: first, there is accurate information on the body of truths it contains; second, there is accurate information on the rise and progress of the subject, the approximate dates in which the epoch-making discoveries were made, the lines and inter-relations of the great discoverers themselves. Little boys, when struggling with the theorem of Pythagoras, find it a great incentive and stimulus to mastery if they are told the facts of the life of Pythagoras with his wanderings from Greece to Sicily sketched out on the map. What one may call the "cricketing" or "bump-

supper" instinct in them rises to enthusiasm when they are told that Pythagoras was in such exultation over the discovery of his proof that he sacrificed an ox to the gods; and I have never yet seen the little boy who, on beholding the elaborate and intricate figure, did not feel that Pythagoras was to be pardoned if he had sacrificed—not a single ox—but a whole herd. Of what tremendous fascination is the combined theory of invariants and geometry, but with what infinite degree of charm is the subject invested when we know the individual contributions of Cayley, Aronhold, Clebsch, Salmon, Hesse—to name only a few of the heroes of the subject—and the psychology of their inter-relations and mutual influence, one on another. Clebsch was turned from applied mathematics to invariant-geometry not by lectures in an academic hall but by a chance conversation at a tea-party. We all recall vividly the Todhunter of our school and student days, and we all have some mental picture—no matter how inaccurate and ill-informed—of the venerable sage. Probably we all agree to some extent in imagining him as a venerable Dr. Dryasdust or placid Don, with a long beard, inhabiting a study which he rarely leaves, and evolving theorems and text-books for conscience' sake. Beyond this veil of darkness and mystery we never dared to penetrate. Does it not add a touch of human interest to the retrospect of our youthful studies to know that the success of the sales of those very text-books over which we used to pore, enabled him to marry an admiral's daughter, and that the combined force of all her charms and the novelty of newly-wedded life were unable to preserve the sanctity of the honeymoon from the unhallowed intrusion of a copy of Hamilton's *Quaternions*. In the days when "First-Courses" on subjects outside the rigid bounds of the curriculum were less frequent than they are to-day, I can remember the thrill with which some of us bought Kelland and Tait's *Quaternions* and learned those ideas on vectors and rotors in space—so different from anything we had heard in the class-room. Before the creative wonders which we pictured ourselves as capable of doing to enlarge this glittering subject, Newton paled into insignificance. But we knew nothing about the great Irish inventor of the Quaternion Analysis, the immortal Sir William Rowan Hamilton. We did not know that he first scratched the symbols of his new calculus with a knife on the parapet of a bridge; that he wrote impassioned love-poetry and meditated suicide on the banks of the Royal Canal for the old tale of unrequited love; that in later, calmer and perhaps saner days he sat in his study at a desk which he approached by a pathway between two piles of paper, like the Israelites going through the Red Sea; and finally, that when these precious papers were removed and edited they were found to be the cemetery of all manner of fish-bones and cutlet remains thrown away or left untouched in utter abstractedness. We might recall the struggles of Monge, the pedlar's son, denied recognition and position in the days when intellect without aristocratic connexion was less valued than it is to-day. Does every lecturer in our engineering and technical colleges, when teaching descriptive geometry, take care to explain how Monge invented the method for military reasons to save time and avoid calculation, and how its use was at first forbidden by the authorities because the required results could be obtained in a shorter time than the Government allowed. The last name I will mention is that of Abel, who first showed that the quintic equation could not be solved by the functions dealt with in our school books, and was thus led to the discovery of those powerful but terrifying engines of analysis, the Abelian Functions. He was a Norwegian mathematician of European fame, for whom no post could be found, and he died in poverty and of semi-starvation at the age of 27. It has been truly said that the world starves its great men and then salves its conscience by erecting statues to them. I end on a hopeful note. At least one university has decided that every candidate for its Higher School Certificate shall exhibit a knowledge of some period of mathematical history. Let us hope that others will follow.

Such is then, in broad outline, a sketch of the meaning and constitution of a school mathematical library. There is undoubtedly a wide-spread feeling among teachers that such a library ought to exist in connexion with every school that is entitled to rank as well-equipped. These vague, floating ideas and aspirations are being crystallised in the form of definite action, and I am glad to say that the Mathematical Association has taken the matter seriously in hand, and at no distant date will issue a suggestive list of books, first in the *Mathematical Gazette*, where the Members of the Association will have opportunities of suggesting additions and modifications, and secondly in pamphlet form, which will be revised from year to year and kept up-to-date. I hope that in the discussion members will state frankly their views and needs in the matter, and so help in the creation of a powerful body of public opinion in favour of such a movement. Empowered and stimulated by such a body of public opinion, the Mathematical Association can go confidently forward in what I believe to be one of the most important and clamant tasks it has ever set itself.

W. P. MILNE.

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**17. Porson and Gulliver.** Porson's instinct for numbers is illustrated by his noting the numerical similarity between the 91 pamphlets, which Swift wrote for 36 factions, and the fourscore-and-eleven chains which were fastened to Gulliver's leg by six-and-thirty padlocks.—Introduction to "A Tale of a Tub," and *Gulliver's Travels*, close of chap. i.

**18.** "There was a time, about the sixth century, . . . when *mathematician* implied *astrologer, conjuror*, and *mathesis* was used to indicate *astrology*. So that the name of our science was in such discredit that a law of Justinian's *Codex* bore the title: *De maleficis et mathematicis et ceteris similibus*, and decreed: *ars autem mathematica damnabilis interdicta est omnino.*

'For an engineer who has to make his way in life the worst he could possibly do is to become educated'—from a little book by H. G. Taylor, Lecturer in C. and Mech. Engineering, King's Coll., University of London: quoted by *The Engineer*.

I hope to get the book: there might be some other startling paradox. (Why does he lecture at all?), and we readers of the *Gazette* ought to be prepared for the designs of those who would have their "artem damnabilem omnino interdictam." The editor rather backs up Taylor and attacks mathematical training himself, mentioning with evident disfavour the time when a certain Technical College was 'a hot bed of mathematics,' and 'a student who did not revel in the higher flights of Conic Sections had a poor chance indeed of achieving distinction.' I rather fancy there is ground for this disfavour and that I have heard private complaints somewhat bearing out the editor's contention."—[*From a letter.*]

**19.** "From Trouville I made the pilgrimage to Beaumont en Auge, a picturesque little town on a hill looking over the sea. '*Ici naquit Laplace*' was the commemorative plaque on a house in the market place. The old building near the church was the military, *i.e.* mathematical and science, college kept by the priests, who discovered Laplace and made him. Sent him to Paris, as too good to be wasted there. But Laplace was a courtier of the court of Napoleon, and turned his back on the good old priests, never went near them again, the snob. Who reads Laplace to-day? His *Mécanique Céleste* was to settle the Universe once for all. I remember d'Ocagne showed me a couple of volumes he had taken away on his holiday, as he felt he must read into them some time or other; there they were unopened, used by his little girl to raise her seat at the piano.

Minchin is old English for nun; so I regret it is too late I didn't christen him Monge, old French for a monk, instead of M.M.—[*From a letter.*]

# NOMOGRAPHY.

By S. Brodetsky, M.A., Ph.D.

The algebraic solution of any problem formulated in mathematical symbols takes one of two forms: it is either given in the explicit form $x=f(a, b, c, ...)$, where $f$ is some functional form of $a, b, c, ...$, which are supposed known in any particular case, or the solution required is to be found from an equation $F(x, a, b, c, ...)=0$, where $F$ is a known functional form of the unknown $x$ and of the quantities $a, b, c, ...$. The mathematician is generally concerned on the one hand with reducing $f(a, b, c, ...)$ into its simplest, often its "prettiest" form, and on the other hand with investigating what method, if any, exists for "solving" the equation $F(x, a, b, c, ...)=0$, *i.e.* finding $x$ in terms of $a, b, c, ...$. The course included in any text-book of algebra consists principally of the former, since the latter is distinctly more difficult.

Algebra in this form is unsatisfactory from two points of view. In practical applications of mathematics we usually need the actual numerical solution of the problem in hand, so that even if a problem has been solved algebraically in the form $x=f(a, b, c, ...)$ the heaviest part of the work is often yet to come in evaluating $f$ with the given values of $a, b, c, ...$. Further, there are many practical problems which are insoluble by ordinary algebraical methods, *i.e.* equations $F(x, a, b\ c, ...)=0$ exist which cannot be solved in the form $x=f(a, b, c, ...)$, yet the numerical value of $x$ is required and must somehow be discovered.

We must depart from the orthodox algebraical processes, and one of the most effective and most popular methods adopted is that of graphs. It is, of course, true that the mathematician's aim in teaching graphs is to introduce the student to the Cartesian geometry, and the promising pupil sees in the method of coordinates a powerful combination and extension of the algebraical and geometrical modes of reasoning. At the same time the use of graphs as a means of obtaining numerical solutions of problems must remain one of their chief claims to the attention and interest of our profession.

The ordinary graphical method is not well adapted for use in all cases where numerical solutions are required. Every school boy of a certain age knows that by means of the intersection of two curves, usually a straight line and some other curve, he can obtain the solution of an otherwise difficult equation. Where he often comes to grief is in the drawing of the graph. And further, apart from the difficulty that so many experience in drawing a curve from its equation, there is another important practical consideration, namely the loss of time involved in plotting a curve each time that a numerical solution is required. It is the object of nomography to make graphical solutions easy and expeditious.

When the solution is given by $x=f(a)$ or $F(x, a)=0$ there is little that one can add to the principle of the ordinary graphical method. The obvious thing to do is to consider $x$ and $a$ as two coordinates (of some convenient system) and plot the curve which exhibits the relation between them. The value of nomography becomes more apparent when $x$ depends on at least two quantities $a, b$.

Let us then suppose that $x=f(a, b)$ or $F(x, a, b)=0$. It shall be our object to indicate the idea of the method of nomography and the way it is applied in a few simple cases. The examples chosen are necessarily such as will be best appreciated by teachers of elementary mathematics.

In Fig. 1 we have three parallel straight lines: the outer ones, $a, b$, are graduated with the same unit, the middle one, $x$, is midway between $a$ and $b$ and is graduated with half their unit. The zeros are all on a line perpendicular to the scales. If now we take the graduation $a$ on scale $a$ and graduation $b$

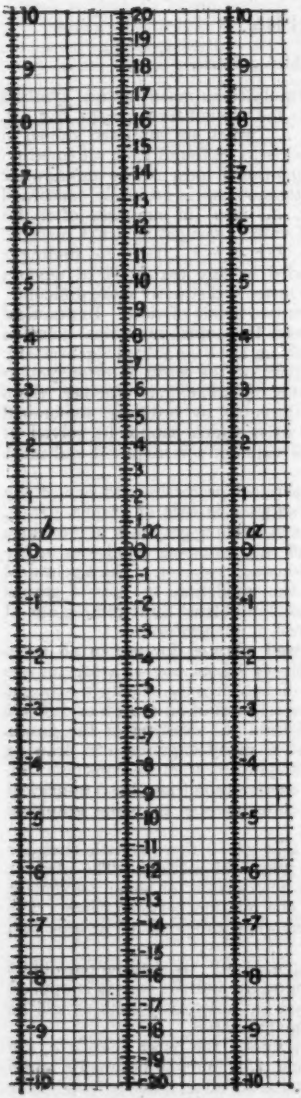


FIG. 1.

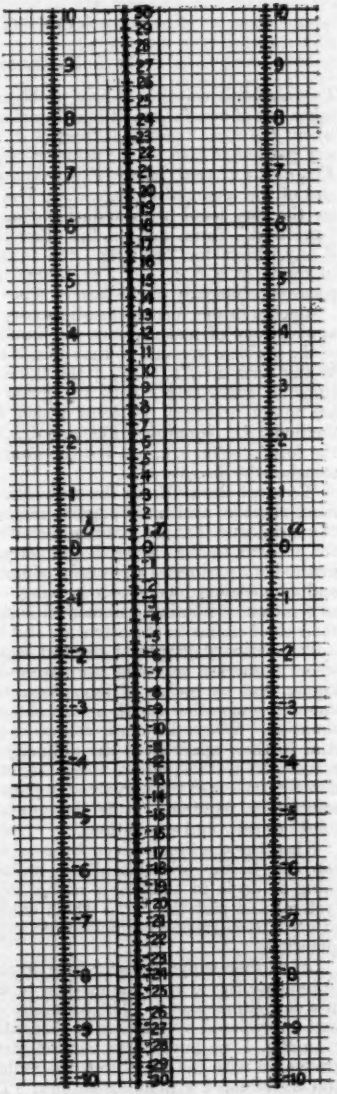


FIG. 2.

on scale $b$, it is clear that the line joining them cuts scale $x$ in the graduation $a+b$. Thus we have in Fig. 1 a nomogram for addition. In general, if any straight line cuts the three scales at graduations $a$, $b$, $x$ we have $x=a+b$ or $x-a=b$, so that Fig. 1 may be used both for addition and for subtraction.

This is the principle of nomography: $x=a+b$ represents a certain type of $f$ in $x=f(a, b)$. For this type or law (*nomos*) the scales in Fig. 1 can be used for all values of $a$ and $b$. There is no restriction on these quantities; but if we have large values of $a$, $b$ we need an inconveniently long figure. This is obviated by taking out a common factor in the form of a multiple of 10. Thus we graduate $a$ and $b$ from $-10$ to $+10$ and $x$ from $-20$ to $+20$, with subdivisions as far as may be possible—in Fig. 1 tenths are given for $a$, $b$ so that hundredths can be estimated, and fifths for $x$ so that fiftieths can be estimated. Then to add up $10a+10b$ we add $a+b$ and multiply $x$ by 10. Similarly, for $10^n a+10^n b$ we add $a+b$ and multiply the result by $10^n$. If we have to add up two excessively minute quantities we write them in the form $10^{-n}a$ and $10^{-n}b$, where $n$ is an integer chosen so that $a$ and $b$ are moderate numbers, and then add $a+b$ and multiply the result by $10^{-n}$.

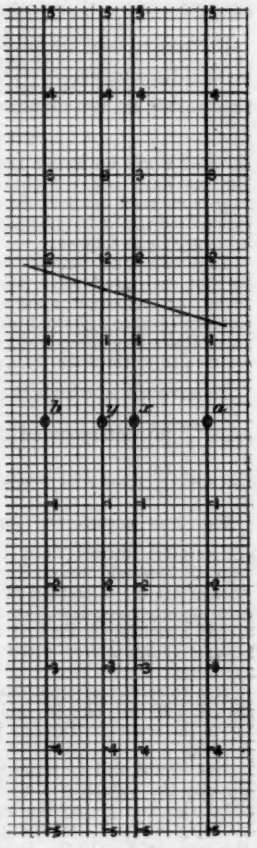

FIG. 3.

Although it is true that the example just given is in itself of no great practical value, its manifold modifications can be of distinct utility. Suppose we wish to find the values of $a+2b$, *i.e.* we desire a nomogram for the type $x=a+2b$ (Fig. 2). Take scales $a$, $b$ as before, graduated with the same unit, but let the third scale $x$ be twice as far from $a$ as it is from $b$, and let its unit of graduation be one-third of the $a$, $b$ unit. The line joining $a$, $b$ cuts the $x$ scale in the graduation $a+2b$, as the reader will easily verify.

More generally, to find the value of $la+mb$ for all values of $a$ and $b$ ($l$, $m$ being given positive quantities), we take $a$, $b$ scales as before and the $x$ scale at distances from $a$, $b$ in the ratio $m/l$, and graduated with $\frac{1}{l+m}$ of the unit used in $a$, $b$. Let us now suppose that the scales $a$, $b$ are fixed once for all in size and position. The $x$ scale will vary in position according to the ratio $m:l$, and its unit will vary also. The latter variation can be eliminated by using the same unit in the $x$ scale as in the $a$, $b$ scales and multiplying the number read off on $x$ by $l+m$. Thus we have three exactly equal parallel scales, the outer ones fixed in position and the inner one having a position dependent on the ratio $m:l$.

Take now any point $x$ $\left(i.e.\text{ the graduation } \frac{x}{l+m}\right)$: any line through it will cut the $a$, $b$ scales at graduations which are such that $la+mb=x$. There are an infinite number of such lines, and, therefore, there are an infinite

number of pairs of values $a$, $b$ satisfying the equation $la+mb=x$. If we can find some way of deciding upon the particular straight line to be used, we shall have unique values of $a$, $b$. This is evidently given by another equation, say $l'a+m'b=y$, where, $l'$, $m'$ are two given positive quantities and $y$ is any given quantity. Let us then introduce a fourth scale, $y$ (Fig. 3), equal in all respects to the $a$, $b$, $x$ scales, but dividing the distance between $a$, $b$ in the ratio $m'/l'$, and let us use the $y$ graduations multiplied by $l'+m'$. Then it is obvious that a line which passes through the point $y$ $\left(\text{i.e. graduation } \frac{y}{l'+m'}\right)$ cuts the scales $a$, $b$ at graduations satisfying the equation $l'a+m'b=y$.

If, then, we take the line through the point $x$ $\left(\text{graduation } \frac{x}{l+m}\right)$ and the point $y$ $\left(\text{graduation } \frac{y}{l'+m'}\right)$, it cuts the scales $a$, $b$ at points whose graduations satisfy the two simultaneous equations

$$la+mb=x, \qquad l'a+m'b=y.$$

In other words we have a graphical solution of these simultaneous equations. But we have not yet obtained a satisfactory method because the scales $x$, $y$ have been specially constructed for the coefficients $l$, $m$; $l'$, $m'$. The generalised method applicable to all positive values of $l$, $m$; $l'$, $m'$ ($x$, $y$ being arbitrary) is as follows:

Take two equal parallel scales $a$, $b$. Graduate the distance between them so that each graduation represents the ratio of its distances from the two scales (*i.e.* construct a so-called segmentary scale). Through a convenient number of these graduations draw lines parallel to $a$, $b$, and construct on them scales like those on $a$, $b$, preferably by means of equidistant lines perpendicular to the scales $a$, $b$, as in squared paper.

To solve the equations $la+mb=x$, $l'a+m'b=y$, take the point $\frac{x}{l+m}$ on the scale $\frac{m}{l}$ (by interpolation at sight if necessary), and the point $\frac{y}{l'+m'}$ on the scale $\frac{m'}{l'}$; the join cuts the $a$, $b$ scales in the solutions required.

Fig. 4 is constructed for positive ratios of $\frac{m}{l}$, $\frac{m'}{l'}$. We can suppose $l$, $l'$ positive always, since if either is negative we can introduce a negative sign all through the corresponding equation. If $m$, $m'$ are both negative change $b$ into $-b$, and the nomogram of Fig. 4 can be used. If one is positive and the other negative, say $m$ positive, $m'$ negative, it will be an easy matter to add a multiple of the first equation to the second so as to make the second assume a form with $m'$ positive. Thus, Fig. 4 can be used for all simultaneous equations.

*E.g.* to solve the equations $3a+8b=23$, $-a+3b=5$, we write the equations $3a+8b=23$, $4a+5b=18$, and take the point $\frac{23}{11}$ on the scale $\frac{8}{3}$ and the point $\frac{18}{9}$ on the scale $\frac{5}{4}$. The line joining the points cuts the $a$, $b$ scales at $a=1{\cdot}73$, $b=2{\cdot}23$.

But Fig. 4 will do more than merely solve for $a$ and $b$. It will give us directly the values of $l''a+m''b$, it being given that $la+mb=x$, $l'a+m'b=y$. Find the line giving the solutions. The value of $l''a+m''b$ is obtained by multiplying by $l''+m''$ the graduation of the intersection of this line with the scale $m''/l''$. If the ratio $m''/l''$ is positive, Fig. 4 can be used at once. If it is negative we can suppose that the expression $l''a+m''b$ is treated in a manner analogous to that mentioned in the case where $m'$ is negative.

Thus, in the example already given, we find that

$$2a+6b=16{\cdot}8 \quad \text{and} \quad 2a-6b=5a+2b-(3a+8b)=-9{\cdot}96.$$

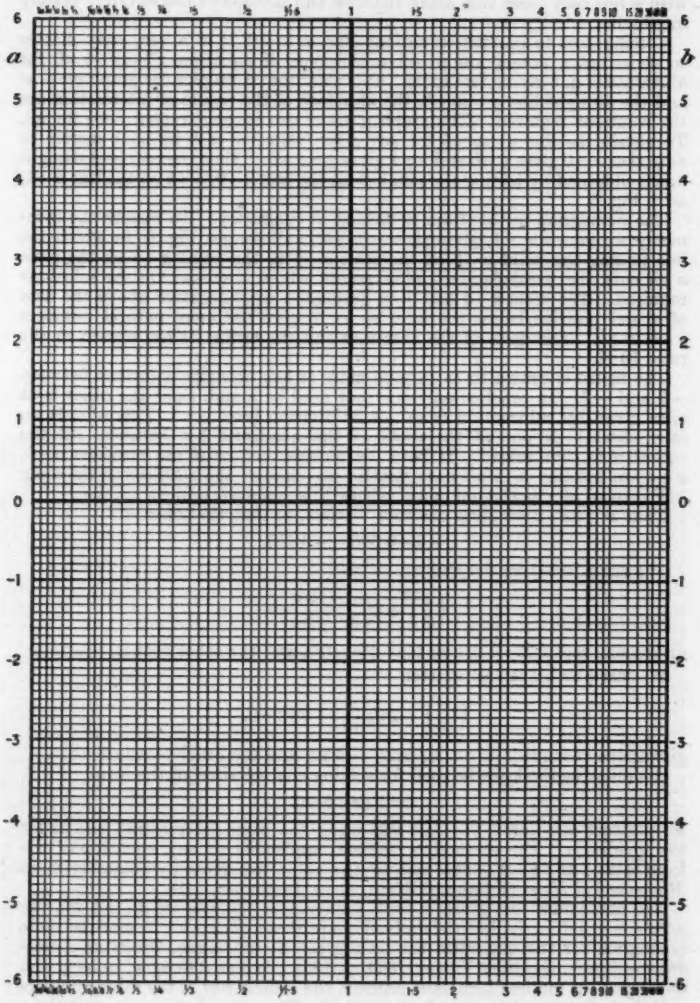


FIG. 4.

The reader will have no difficulty in constructing nomograms for multiplication and division, and more generally for products $a^l b^m$, by the use of logarithmic instead of regular scales, as in the slide rule. A little practice

with a few easy cases will make the idea and method of nomography readily intelligible.

Coming now to implicit equations $F(x, a, b)=0$, it is a very easy matter to construct a nomogram to solve the type of equation $x^2+ax+b=0$ for all values of $a$, $b$ giving real roots. It is clear that if we plot the graph $y=-x^2$ and the straight line $y=ax+b$, the abscissae of the points of intersection are the solutions required. The graph $y=-x^2$ can be constructed once for all. To obtain the line $y=ax+b$ for any given values of $a$, $b$ we have merely to note that $x=0$ gives $y=b$ and $x=1$ gives $y=a+b$. Thus the required straight line is obtained by joining the points $b$ on the line $x=0$, $a+b$ on the line $x=1$, as in Fig. 5.

The nomogram in this form is lacking in symmetry. To restore the symmetry we take $x=-1$, giving $y=-(a-b)$, and we get Fig. 6, in which the line joining the point $a-b$ on the scale $a-b$ to the point $a+b$ on the scale $a+b$ cuts the parabola $y=-x^2$ in points whose abscissae are the solutions required. The scale $a-b$ has been graduated with negative above the axis of $x$ and positive below the axis, so as to give the more natural numbers $a-b$, $a+b$. The units of the abscissae and the ordinates are chosen in the ratio 10 : 1.

The figure is drawn so as to give more or less accurate solutions between $+2$ and $-2$. In practice only one root need be found, since the other is at once given by subtraction from $-a$ or division into $+b$. (This affords an easy check.) If one of the roots of the quadratic lies between $\pm 2$ the equation is thus solved. If both lie outside $\pm 2$, or if the numbers $a-b$. $a+b$ are too big, it is an easy matter to change the coefficients by using $x/N$ instead of $x$ as the unknown, where $N$ is some convenient number.

Thus $x^2+4x+2=0$ gives one root $-0{\cdot}585$, so that the other is

$$-4+0{\cdot}585=-3{\cdot}415.$$

$x^2+12x+10=0$ can be written

$$\left(\frac{x}{3}\right)^2+4\left(\frac{x}{3}\right)+\frac{10}{9}=0,$$

giving one root $x/3=-0{\cdot}3$, so that $x=-0{\cdot}9$, and the other root is $=-11{\cdot}1$.

$x^2-29x+53=0$ can be written

$$\left(\frac{x}{10}\right)^2-2{\cdot}9\left(\frac{x}{10}\right)+0{\cdot}53=0,$$

giving one root $x/10=0{\cdot}195$, so that $x=1{\cdot}95$, and the other root is $27{\cdot}05$.

A similar method can be used for solving any equation of the form

$$x^n+ax+b=0$$

by plotting the curve $y=-x^n$ and using the same intersection line as for the quadratic. The method is, in fact, applicable to any equation of the form $\phi(x)+ax+b=0$. The case of the cubic was given by Sommerville in *Mathematical Notes* (Edinburgh Math. Soc.), December, 1916.

The nomogram just given is still inconvenient because we do not use the numbers $a$, $b$ directly, but the derived quantities $a-b$, $a+b$. A more suitable nomogram for the quadratic has been given by d'Ocagne, *Traité de Nomographie* (Paris, Gauthier Villars, 1899, p. 183). It was d'Ocagne, in fact, who developed the method here used, namely, the collinearity of the three numbers $x$, $a$, $b$ on three scales suitably graduated. The general theory is that three curves are found, $x$, $a$, $b$, graduated in such a way that the line joining the points $a$, $b$ gives the corresponding value $x$ by its intersection with the $x$ curve. In practice the scales $a$, $b$ are straight, and $x$ is curved if necessary. It was d'Ocagne who introduced the simplification of parallel coordinates, namely, measuring $a$, $b$ directly on parallel equal scales.

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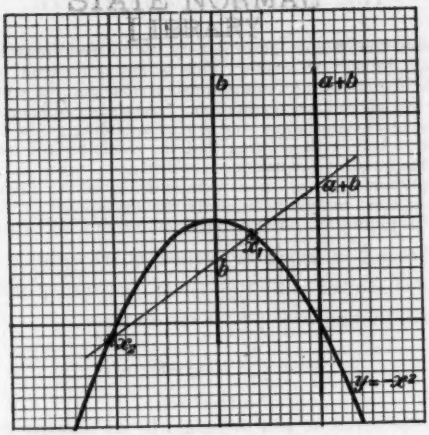


FIG. 5.

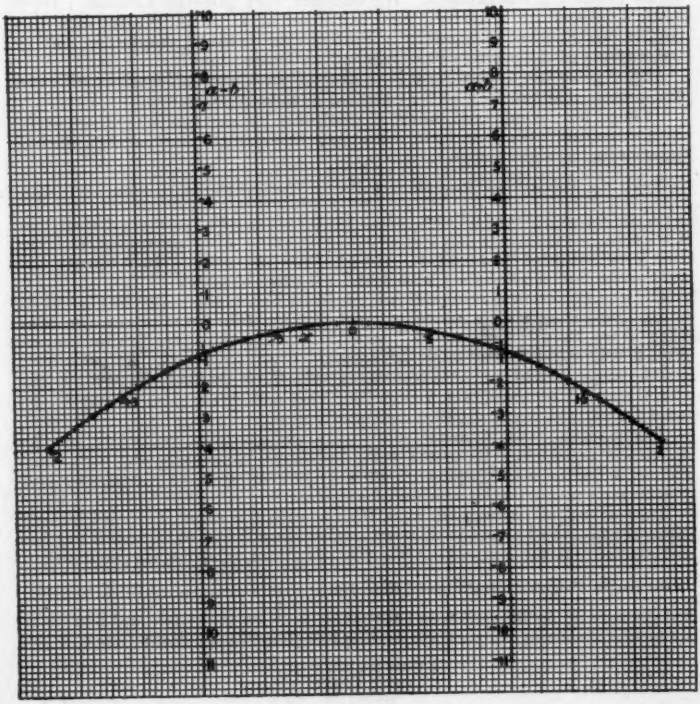


FIG. 6.

It is often said that a nomogram is very useful when once it is made, but that it is very difficult to make. This is not quite true. In d'Ocagne's treatise will be found full and simple instructions for the construction of a nomogram for any given type of equation. The theory of his method involves the use of tangential coordinates, and is correspondingly powerful. We shall content ourselves here with showing that the result arrived at by d'Ocagne, in the case of the quadratic equation, is correct.

Plot the hyperbola $\xi(\xi+\eta)+\eta=0$ with suitable units. Graduate the axis of $\eta$; call it scale $b$. Graduate similarly the asymptote $\xi=-1$, and call it scale $a$. The line joining the point $a$ to the point $b$ cuts the hyperbola in points which are such that $\frac{\eta}{\xi}=x$, giving the solutions of the equation $x^2+ax+b=0$.

The equation of the line is $\eta=b(\xi+1)-a\xi$, and that of the hyperbola is $\eta=-\frac{\xi^2}{\xi+1}$. Substituting $x=-\frac{\xi}{\xi+1}$ in the equation

$$\eta=-\frac{\xi^2}{\xi+1}=b(\xi+1)-a\xi,$$

we get

$$x^2+ax+b=0.$$

To graduate the curve we join the origin to the negative graduations on the asymptote, and the points of intersection with the hyperbola have the equal positive graduations $x$. We consider only positive solutions; for if both solutions are negative, use $-x$ instead of $x$, *i.e.* change the sign of $a$, and if one solution is negative it need not be obtained graphically but by subtraction of the positive root from $-a$ or division into $b$.

Fig. 7 gives solutions from 0 to 9, and will be found sufficient for all cases by suitable manipulation of the equation as already explained.

It must not be supposed, however, that only parallel coordinates are useful in nomograms. Scales $a$, $b$ can be on any two straight lines, as *e.g.* on lines intersecting at right angles.

Take *e.g.* the optical formula $\frac{1}{a}+\frac{1}{b}=\frac{1}{x}$. To find the third quantity, given two, we can proceed as follows. Join the point $a$ on the $x$-axis of rectangular Cartesian coordinates to the point $b$ on the $y$-axis. The intersection with the line $y=x$ has $x$ for its abscissa. We graduate the $x$ and $y$ axes equally and uniformly, and the line $y=x$ with a unit $\sqrt{2}$ times as great.

Oblique Cartesian coordinates are generally taboo, especially at an Association consisting mainly of schoolmasters. It will, nevertheless, be clear that if we have "rhombussed" instead of squared paper, the method of Fig. 8 will apply to any angle between the coordinate axes. In particular, let the angle be 120°. The advantage is that the $x$ graduations now become exactly equal to those of $a$, $b$ (Fig. 9). This nomogram can, of course, be used for all positive or negative values of $a$, $b$, $x$.

A nomogram with perpendicular scales was given by Whittaker for the quadratic equation. It shares with d'Ocagne's nomogram the advantage that $a$, $b$ can be used directly on suitably graduated scales. The theory of Whittaker's nomogram can be stated as follows:

Take a pair of rectangular axes, $O\xi$, $O\eta$ with the same unit, and draw a circle of radius half a unit, with its centre $C$ at the point $(\frac{1}{2}, 0)$, so that it touches the axis of $\eta$ at the origin, $O$ in Fig. 10. If $P$ is a point on the circle and $\theta$ is the angle $PC\xi$, the coordinates of $P$ are $\left(\frac{1+\cos\theta}{2}, \frac{\sin\theta}{2}\right)$. But if we take the point $(\alpha, 0)$ on the $\xi$-axis and the point $(0, \beta)$ on the $\eta$-axis and join them by means of a straight line, then the line will pass through $P$ if we have

$$\frac{1+\cos\theta}{2\alpha}+\frac{\sin\theta}{2\beta}=1.$$

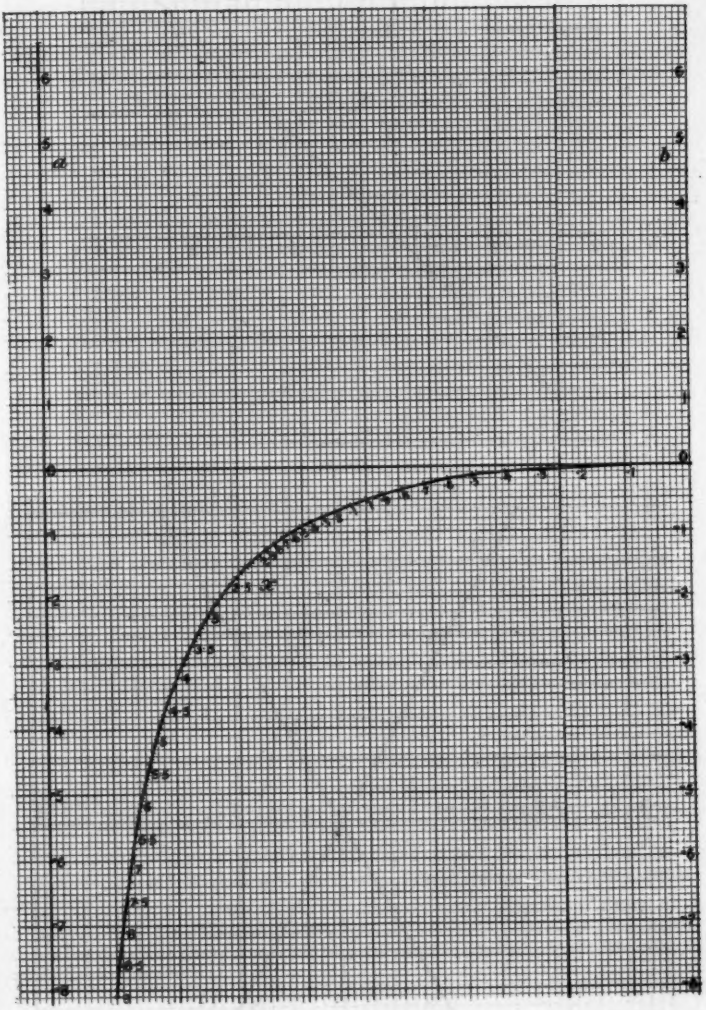

FIG. 7.

Put $x = \tan\frac{\theta}{2}$, then we obtain the equation

$$\frac{1}{(1+x^2)\alpha} + \frac{x}{(1+x^2)\beta} = 1,$$

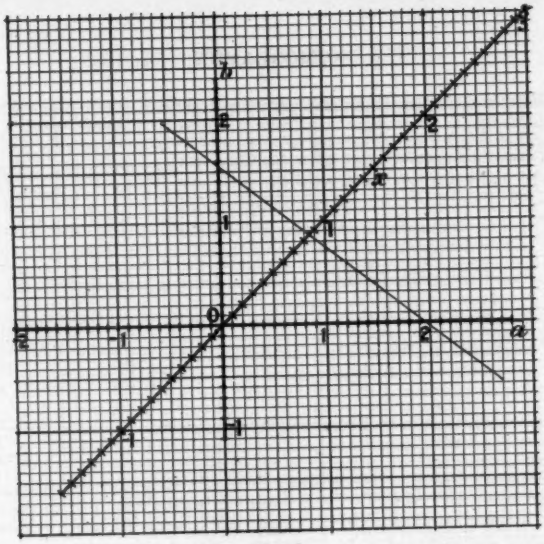


FIG. 8.

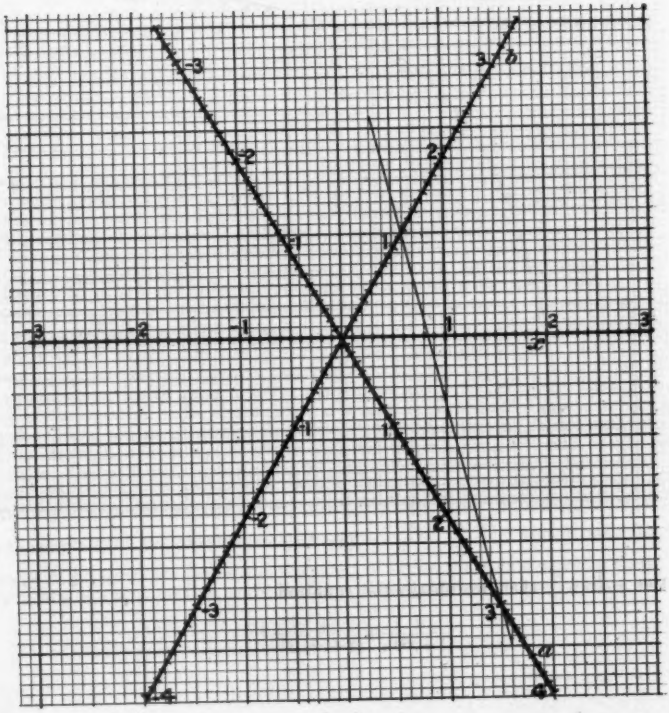


FIG. 9.

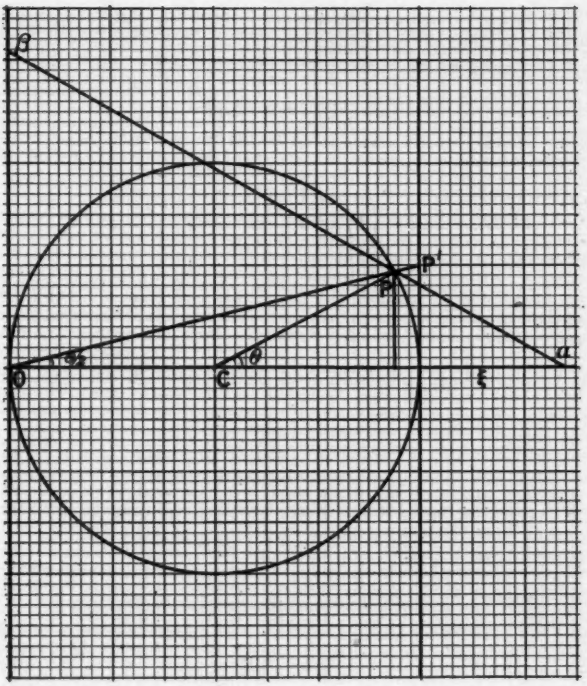


FIG. 10.

*i.e.* $x$ satisfies the quadratic equation $x^2-\frac{x}{\beta}+1-\frac{1}{a}=0$. If we join $OP$ and produce it to cut the line $\xi=1$ in $P'$, the ordinate of $P'$ is a solution of the quadratic equation $x^2+ax+b=0$; $a$, $\beta$ being so chosen that $\beta=-\frac{1}{a}$, $a=\frac{1}{1-b}$. Hence, graduate the axis of $\xi$ according to the law $\frac{1}{1-b}$, and the axis of $\eta$ according to the law $+\frac{1}{a}$, *using the same unit*, and graduate the circumference of the circle by means of lines joining the origin to a uniform scale, with the same unit, on the line $\xi=1$. Then the line joining $-a$ on $\eta$ to $b$ on $\xi$ cuts the circle at points whose graduations are the roots of the given quadratic.

In Fig. 11 only half the circle is drawn, since we need only find a positive root.

The great advantage of Whittaker's nomogram is the ease with which it can be constructed, since the curve to be drawn is a circle and the graduations are given by simple geometry. On the other hand the complication of the graduations on the two scales $a$, $b$ is a distinct disadvantage.

It is hoped that enough has now been said to make it clear that the nomogram represents a great advance on the ordinary graphical methods, and that

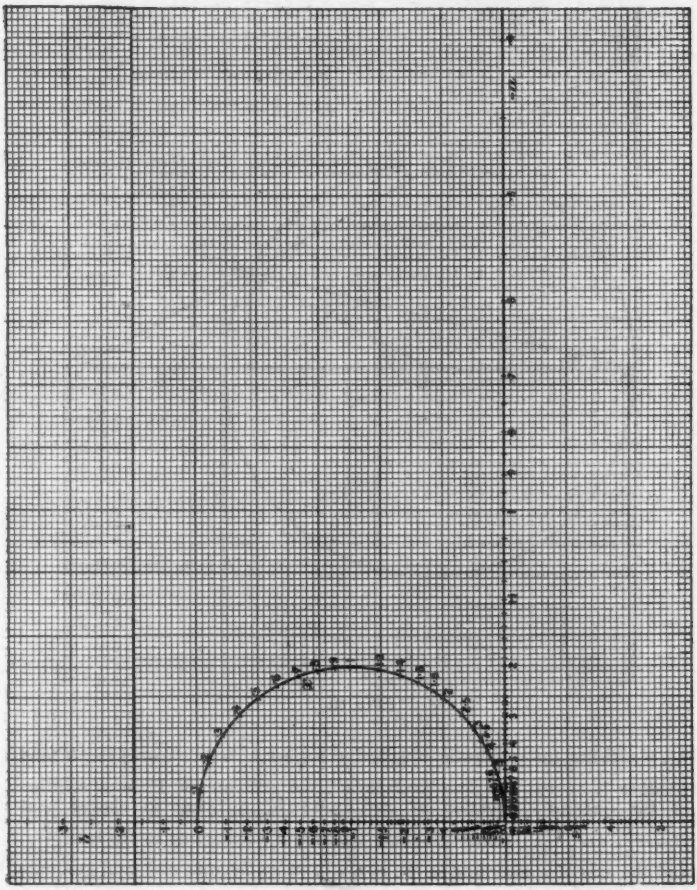

FIG. 11.

its construction for a given type of relation between $x$ and two quantities $a$, $b$ is simple enough in many cases, both in theory and in execution. Nomograms are used in innumerable branches of technology, such as railway engineering, hydraulics, trench digging, and various kinds of military operations. In this short account of the subject our aim has been to illustrate principles rather than to enumerate cases of actual practical application. For these, readers are advised to read d'Ocagne's fascinating book and the various authorities referred to by him.

S. BRODETSKY.

# SOME SUGGESTIONS FOR A PRESENTMENT OF MATHEMATICS IN CLOSER TOUCH WITH REALITY.

By G. Goodwill, B.Sc.

In spite of the achievements of mathematics, and of a very general recognition of the fact that mathematics is essential to the progress of knowledge in most of its departments, there exists a certain amount of contempt for the symbols, the formulae, and the abstractions of the subject. For a generation we have struggled against this feeling, and have endeavoured to correct the defects to which the feeling is due, but I do not think that many are satisfied with the success of our efforts. Those who have attended to the widespread demands for a more generous and thorough provision for education, as representing the only sure foundation for national efficiency and progress, have remarked with dismay the little interest that is shown in mathematical education. Surely this state of things demands, at the present time, our most urgent attention.

Why have our efforts not been more successful? I believe that with a more robust faith in our subject, we should have been more successful. We have been too ready to acquiesce in the contempt for symbols and formulae and abstractions, and we have rushed, sometimes with indecent haste, to clothe what seemed the nakedness of our subject with concrete illustrations and applications, dragged in from all sorts of directions. The result has been a loss of unity of purpose; and while there has been a gain of interest in our lessons, the gain has not been a gain in mathematical interest, but in a hundred and one miscellaneous topics, which has served rather to complicate and overload our work, and to divert the attention and interest from mathematics itself.

I believe that we have not succeeded, because we have begun at the wrong end, in taking the contents of our mathematical syllabuses and trying to find illustrations and applications. Mathematical thought does not originate, and never did originate, in this way. It originated in the attempt to understand our experiences, and in the attempt to formulate our knowledge with clearness and precision.

The means which mathematics employs to this end is not peculiar to itself.

In seeking to understand any kind of behaviour, we endeavour by a process of abstraction to discover some principle which will explain it. But the abstracted principle remains shadowy and elusive, unless by some means it is clearly characterised. This characterisation is most strikingly effected by the use of a symbol. Loyalty by the flag, devotion and sacrifice by the cross, oppression by the iron heel.

It is only when symbols and formulae are presented in the first instance, detached from those experiences for the representation of which they have been devised, that they seem empty and trivial, indeed contemptible in their meaninglessness.

Now the endeavour to know and to understand represents the aim of science in general, but to devise a system of symbolic representation by means of which this knowledge is given precise formulation, and which gives currency to our ideas in thought and expression—this is the special function of mathematics.

In a nut-shell, then, my prescription for remedying the defects to which I have alluded is this. In teaching elementary mathematics we should aim at our pupils acquiring a precisely formulated knowledge and understanding of certain groups of phenomena with which their experience has made them acquainted, or to which we can readily introduce them; and I submit that in pursuing this process they will learn mathematics more surely than in any other way. I contend that in the consideration of some

well chosen range of experience, a natural field for the exercise of a true mathematical faculty is found, and that only in this way can a sufficient appreciation of the technique of the subject be gained. Moreover, the sort of mathematical expert which is produced by confining the attention of our pupils to the rules and technicalities of mathematics by themselves, is the sort of person who has brought the subject into disrepute.

Now, in order to present our subject in the way for which I contend, it is necessary for us to claim for our own those branches of knowledge which have become mathematical. I believe that in the case of any subject, or well defined range of a subject, the ideas of which have been formulated mathematically, the teaching of that subject in particular would gain, and the teaching of mathematics would gain, if it were taught as part of mathematics. School mathematics already claims a good many such—money affairs, geometry, mensuration, mechanics. I would go further and claim much of what is taught as part of geography, optics, magnetism and electricity, as well as astronomy. I contend that in our mathematical lessons our aim should be for our pupils to gain a mathematical knowledge of a suitable range of several of these subjects.

In process of dealing with an individual subject suitable arithmetical expressions, symbols and formulae, diagrams and graphical representations would be devised and manipulated.

In this way mathematics provides us with a sort of symbolic structure, which is the counterpart of the realities of that range of experience which has been under consideration. Several subjects must be taken and dealt with. Practice in mathematical technique would not be stinted, but would be relevant to the work in hand; and there would be some interest in enquiring what was the meaning of modifications which might be suggested, of the formulae and graphs which had been devised. At the same time it would be noticed that graphs and formulae of the same shape would make their appearance in different subjects, and in this way a legitimate interest in functions of different forms would be enlisted.

The unity of mathematics would in the first instance be exhibited in its appearance as a unified symbolic structure, recalling those ideas of the phenomena which its symbolism had been devised to represent, this unity depending on the relations between the phenomena within the particular range of experience which had been studied. Gradually a unity in mathematics would be generated that was purely abstract, *i.e.* independent of the phenomena which its symbolism represented, a unity wider and more and more comprehensive from the fact that the symbols and expressions at out disposal are found to be applicable to different ranges of phenomena, so that the ideal mathematical structure exhibits a generalised scheme of behaviour in regard to phenomena which apparently had nothing in common.

In order to carry out such a presentment of mathematics as this, the subjects of our mathematical curriculum would have to include a list of something of the following sort: Money affairs, geometry, mensuration, mechanics, astronomy, optics, heat, magnetism and electricity. We should require syllabuses in these subjects limited to such a range as could be honestly dealt with on strictly mathematical lines. Arithmetic, algebra, trigonometry and the calculus would be learned incidentally to these subjects. Textbooks in these latter subjects would be required, but they would be compact and frankly technical in character, providing examples for drill, but excluding discursive illustrations as well as those large collections of miscellaneous applications now usually given.

Some years ago it was my privilege to carry out a scheme of Mr. W. C. Fletcher's for the teaching of very elementary experimental optics on strictly mathematical lines, embracing of course the formulation and application to simple cases of the laws of reflexion and refraction, with their more immediate

consequences. Of the mathematical value of such a course I believe there can be no shadow of doubt. Later I introduced the course to a mathematical colleague elsewhere, and after working through it he told me he had never had such a satisfactory scheme to work to.

During the last three years I have had the opportunity of putting into practice a scheme on similar strictly mathematical lines, for the teaching of experimental mechanics, and in spite of the criticism with which the scheme had met in some quarters, to the effect that it was far too logical and difficult for young boys, the experiment has met with a measure of success which has far exceeded my expectations.

I am quite sure that while work of this kind is going on, there is some real appreciation on the part of the boys of the fact that while on the one hand we are gaining precise ideas about mechanical behaviour, on the other hand we are elaborating a definite mathematical structure, which is the counterpart of the range of experience to which we have attended.

This parallelism between the realities of experience on the one hand, and the symbolic mathematical structure on the other, is the essence of the presentment of mathematics for which I contend; and to whatever extent the pupils may appreciate this parallelism, there is no doubt as to the interest, indeed the enthusiasm, for the purely mathematical work involved, even in the case of classes of quite poor capacity.

Without having clearly recognised this as a definite method of presentment, we already to a considerable extent fulfil its requirements, and we are doing so much more than we did. Arithmetic is taught largely in connection with money affairs and with mensuration. Now, however, in the mathematical syllabus of the Girls' School Committee, the arithmetic of citizenship is set out as a definite subject for study. This is a development of exactly the sort for which I contend.

If we deal mathematically with suitably limited ranges of science subjects, the correlation of mathematics and science teaching will be provided for. Our encroachment on the domain of science teaching would not be serious. While the work of the science master would remain predominantly investigational and experimental, aiming principally at the acquisition of knowledge, ours would consist mainly in giving precision to, and formulating, knowledge which was readier to hand, or in some degree already known intuitively, requiring only occasional simple experiments. We should also in certain cases accept the results of the work of the science master as the assumptions on which our mathematical formulation was based.

There is, of course, much scientific knowledge of which the intelligent citizen should be put into possession, which is not susceptible of a purely mathematical formulation suited to the school boy. Scientific knowledge within this range should be claimed by the science master. On the other hand it is vital to mathematics that it should claim to deal with those subjects or ranges of subjects which are capable of mathematical formulation with the ability that the normal school boy can reasonably be expected to bring to his work. With such an understanding between us, the science master would not find his pupils wanting in the mathematical ability to pursue his work as far afield as he might reasonably require, while we should not be at a loss for those general ideas of which our mathematical structure is the symbol.

It should not be necessary for me to contend that the interests of pure mathematics, in the most strict sense of the term, would be most generously served by such a presentment of the subject. Once the mathematical structure is begun, and exists as a vital structure, its growth and development at all points is inevitable. We all know, as an historical fact, that pure mathematics has originated, and has renewed its vitality, in this way. We know that it has been renewed from time to time by the discovery that

elements in it which had been regarded as purely imaginary, were no more imaginary than the rest, but were in fact the symbolic counterpart of aspects of reality, which, without the aid of pure mathematics, would not have been suspected.

Such a presentment of mathematics would involve, except in the case of specialists, the cutting out of much of the consideration now given to detail in arithmetic, algebra and trigonometry. A thorough and sound ability to multiply and divide decimals would be essential, but with the cutting out of masses of miscellaneous applications, more attention could perhaps be given to the theory of numbers. Some dexterity in the manipulation of factors, fractions, and different kinds of equations would have to be sacrificed. In numerical trigonometry, I find pupils can understand the subject best if I begin with the examples which are usually found near the end of the text-books, and I believe that the presentment would gain in clearness if three-fourths of the earlier portions of these books were cut out. In geometry other curves beside the circle would be examined, but the detail in which rectilinear figures and the circle and other figures were studied would depend on the requirements of the mathematical curriculum as a whole. The principle would be, ruthlessly to exclude detail for which there was not a call arising naturally from the mathematical curiosity of the pupil, or from the need of giving precision to facts which for any reason are interesting or important.

G. GOODWILL.

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**20.** "Geometrical communications to scientific journals are not so common as they used to be, which may partly be attributed to the expense of woodcuts, and partly to the decline of taste for geometry. The latter is accelerated by the paucity of geometrical reading arising from the former cause: and its effect upon soundness of mathematical habits is deadly. But there is no reason why the expense of woodcuts should place any obstacle in the way. I am satisfied, from sufficient trial, that when proper description of the diagram is given in the test, the person who draws his own diagram from the text will arrive at the author's meaning in half the time which is employed by another, to whom the successive appearance of the parts is prevented by his seeing the whole from the beginning. The great work of M. Chasles has not a single diagram.

It will be asked whether the omission of diagrams will not cause a misprint (such as may sometimes occur) to be fatal to the reader's chance of arriving at the result. Putting out of view the very great confusion which a misprint causes when it shows a variance with the diagram, and leaves the reader unable to decide which is wrong, I will remark that it is possible very greatly to diminish the risk of error in the description of the diagram. This is all that is wanted; for grant the reader once able to establish the diagram, and his position is then as good as (at least, I say better than) that of the person who has had the use of a woodcut. My plan would be *to double every letter the first time it occurs*. Instead of 'Let $AB$ be a given straight line, and $C$ a point without it,' I should write 'Let $AABB$ be a given straight line, and $CC$ a point without it.' This would present, I think, a twofold advantage. First, neither author nor compositor is so likely to put in $DD$ instead of $BB$ as $D$ instead of $B$. Secondly in looking back to recover the meaning of a letter, the eye would be caught by the reduplication of which marks its first appearance, whereas at present search is often necessary. I need not state that the omission of the diagram would compel writers to be complete: at present they sometimes (but always implying profession of the contrary) rely upon it that the reader will supply omissions by the diagram."—*Phil. Mag.* and *J. of Sci.* XXXV. Sept. 1849, pp. 165-6. *A. De Morgan.*

## MATHEMATICAL NOTES.

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Roll a piece of paper round a circular cylinder, and on it, by means of a compass used in the ordinary way, describe a closed curve.

When the paper is removed and placed on a flat surface this curve is seen to be an oval with two axes of symmetry.

Taking these as axes of coordinates, the equation of the curve is found without difficulty to be

$$\sqrt{b^2-y^2}=a\sin\frac{x}{a},$$

or

$$y^2=b^2-a^2\sin^2\frac{x}{a},$$

where $a$ = diameter of cylinder
and $b$ = the "radius" with which the curve was described.

On putting $b=a$, we get $\quad y=\pm a\cos\frac{x}{a}$,

*i.e.* the "curve of cosines" and its reflexion in the axis of $x$.

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elements in it which had been regarded as purely imaginary, were no more imaginary than the rest, but were in fact the symbolic counterpart of aspects of reality, which, without the aid of pure mathematics, would not have been suspected.

Such a presentment of mathematics would involve, except in the case of specialists, the cutting out of much of the consideration now given to detail in arithmetic, algebra and trigonometry. A thorough and sound ability to multiply and divide decimals would be essential, but with the cutting out of masses of miscellaneous applications, more attention could perhaps be given to the theory of numbers. Some dexterity in the manipulation of factors, fractions, and different kinds of equations would have to be sacrificed. In numerical trigonometry, I find pupils can understand the subject best if I begin with the examples which are usually found near the end of the text-books, and I believe that the presentment would gain in clearness if three-fourths of the earlier portions of these books were cut out. In geometry other curves beside the circle would be examined, but the detail in which rectilinear figures and the circle and other figures were studied would depend on the requirements of the mathematical curriculum as a whole. The principle would be, ruthlessly to exclude detail for which there was not a call arising naturally from the mathematical curiosity of the pupil, or from the need of giving precision to facts which for any reason are interesting or important.

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taneously. This is effected by projecting the two points into the circular points in the plane of projection. The projections of the conics will now go through the circular points in the new plane, and so they are all circles. The student of course understands that such a projection is an imaginary one."

This is hardly convincing. From the preface to the first edition it appears that Dr. Askwith, like other authors, relies on the "Principle of Continuity" for the introduction of the "Imaginary," although this principle does not seem to be clearly stated or explained. The Principle of Continuity may be regarded as an expression of the fact—confirmed by von Staudt's work—that imaginary points and lines have many projective properties in common with the real points and lines. That their properties are not identical is shown by Dr. Askwith in Article 244. It is dangerous, without full explanation, to present any proof depending on the statement that a pair of real points can, by an imaginary projection, be projected into a pair of conjugate imaginary points or *vice versâ*. Such a projection converts, with few exceptions, all real points in one real plane into imaginary points in another real plane. The interesting results given in the chapter in question are well worthy of the attention of the student, but it might be better to present them as illustrations of the Principle of Continuity rather than to infer that they can be "established" by projecting real into imaginary points. This may seem a trifling matter, but those who object to projective methods largely base their objection on the practice of projecting real points into the circular points at infinity.

Dr. Askwith's book is well printed in the best style of the Cambridge University Press. The figures are good, and it can be unreservedly recommended. It supplies a real want, and should materially contribute towards the advancement of the knowledge of Pure Geometry. J. L. S. H.

## THE LIBRARY.

### Change of Address.

The Library is now at 9 Brunswick Square, W.C., the new premises of the Teachers' Guild.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

A copy of Prof. A. N. Whitehead's *Organisation of Thought* has been added to the Library.

### Scarce Back Numbers.

Reserves are kept of A.I.G.T. Reports and Gazettes, and, from time to time, orders come for sets of these. We are now unable to fulfil such orders for want of certain back numbers, which the Librarian will be glad to buy from any member who can spare them, or to exchange other back numbers for them:

Gazette No. 8 (very important).
A.I.G.T. Report No. 11 (very important).
A.I.G.T. Reports, Nos. 10, 12.

## ERRATA.

p. 172. Equation (17), read $c=0{\cdot}6733$.
p. 173. Equation (19), for last $dx$ read $d\theta$.

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